On the non-equilibrium fluctuations of partial exclusion with open boundary

Patricia Gonçalves

Inhomogeneous Random Systems

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The plan for today's lecture:

1. Exclusion (ONE particle per site): hydrodynamics and fluctuations
2. Partial exclusion (α particles per site)
3. Correlation estimates

Joint with

Chiara Franceschini, Milton Jara, Beatriz Salvador


The dynamics:

- For $N \geq 1$ let $\Lambda_N = \{1, \ldots, N-1\}$.
- We denote the process by $\{\eta_t : t \geq 0\}$ which has state space $\Omega_N := \{0,1\}^{\Lambda_N}$.
- The infinitesimal generator $\mathscr{L}_N = \mathscr{L}_{N,0} + \mathscr{L}_{N,b}$ is given on $f : \Omega_N \to \mathbb{R}$, by

$$(\mathscr{L}_{N,0} f)(\eta) = \sum_{x=1}^{N-2} c_{x,x+1}(\eta)\Big(f(\eta^{x,x+1}) - f(\eta)\Big),$$

$$(\mathscr{L}_{N,b} f)(\eta) = \frac{\kappa}{N^\theta} \sum_{x \in \{1,N-1\}} c_{r_x}(\eta(x))\Big(f(\eta^{x}) - f(\eta)\Big),$$

where $c_{x,x+1}(\eta) := \eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))$,
for $x = 1$ and $x = N-1$,
$c_{r_x}(\eta(x)) = r_x(1-\eta(x)) + (1-r_x)\eta(x)$,
$r_1 = \alpha$ and $r_{N-1} = \beta$.

Invariant measures:

- If $\alpha = \beta = \rho$ the Bernoulli product measures are invariant (equilibrium measures): $\nu_\rho(\eta : \eta(x) = 1) = \rho$.

- If $\alpha \neq \beta$ the Bernoulli product measure is no longer invariant, but since we have a finite state irreducible Markov process there exists a UNIQUE invariant measure: the stationary measure (non-equilibrium) denoted by μ_{ss}.

- By the matrix ansatz method one can get information about this measure. (Not for the partial exclusion!)

Hydrodynamics

$\mathscr{H}ydrodynamic\ \mathscr{L}imit$:

♣ For $\eta \in \Omega_N$, let $\pi_t^N(\eta, dq) = \frac{1}{N}\sum_{x=1}^{N-1} \eta_{tN^2}(x)\delta_{x/N}(dq)$, be the *empirical measure. (Diffusive time scaling!)*

♣ Assumption: fix $g : [0,1] \to [0,1]$ measurable and probability measures $\{\mu_N\}_{N\geq 1}$ such that for every $H \in C([0,1])$,

$$\frac{1}{N}\sum_{x=1}^{N-1} H\left(\tfrac{x}{N}\right)\eta(x) \to_{N\to+\infty} \int_0^1 H(q)\, g(q)dq,$$

wrt μ_N. (μ_N is associated to $g(\cdot)$)

♣ Then: for any $t > 0$,

$$\pi_t^N(\eta, dq) \to_{N\to+\infty} \rho(t,q)dq,$$

wrt $\mu_N(t)$, where $\rho(t,q)$ evolves according to a PDE, the hydrodynamic equation.

$\mathscr{H}ydrodynamic\ \mathscr{L}imit$:

Theorem [Baldasso et al]:

Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta_{tN^2}(x) - \int_0^1 H(q)\rho(t,q)dq\Big| > \delta\Big) = 0$$

and $\rho_t(\cdot)$ is the UNIQUE weak solution of the heat equation $\partial_t \rho_t(q) = \partial_q^2 \rho_t(q)$ with

- $\theta > 1$ **Neumann b.c.:** $\partial_q \rho_t(0) = \partial_q \rho_t(1) = 0$.
- $\theta = 1$ **Robin b.c.:**
  $\partial_q \rho_t(0) = \kappa(\rho_t(0) - \alpha), \qquad \partial_q \rho_t(1) = \kappa(\beta - \rho_t(1))$.
- $\theta < 1$ **Dirichlet b.c.:** $\rho_t(0) = \alpha$, $\rho_t(1) = \beta$.

Fluctuations

$\mathcal{N}$*on-equilibrium fluctuations* $(\theta \geq 0)$:

Definition (Density fluctuation field)

The density fluctuation field $\mathcal{Y}_{\cdot}^{N}$ is the time-trajectory of linear functionals acting on functions $H \in \mathcal{S}_\theta$ as

$$\mathcal{Y}_t^N(H) \ = \ \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} H\big(\tfrac{x}{N}\big)\Big(\eta_{tN^2}(x) - \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]\Big)\,.$$

Definition (The space of test functions)

Let $\mathcal{S}_\theta$ denote the set of functions $H \in C^\infty([0,1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds that

1. for $\theta < 1$: $\partial_u^{2k} H(0) = \partial_u^{2k} H(1) = 0$;
2. for $\theta = 1$:
   $\partial_u^{2k+1} H(0) = \partial_u^{2k} H(0), \qquad \partial_u^{2k+1} H(1) = -\partial_u^{2k} H(1)$;
3. for $\theta > 1$: $\partial_u^{2k+1} H(0) = \partial_u^{2k+1} H(1) = 0$.

$\mathscr{F}luctuations$: $\theta = 1$

What are the conditions on the initial state μ_N?:

- For each $N \in \mathbb{N}$, the measure μ_N is associated to a measurable profile $\rho_0 : [0,1] \to [0,1]$ (This is the same condition for hydrodynamics!).
- For $\rho_0^N(x) = \mathbb{E}_{\mu_N}[\eta_0(x)]$

$$\max_{x \in \Lambda_N} \left| \rho_0^N(x) - \rho_0(\tfrac{x}{N}) \right| \lesssim \frac{1}{N}.$$

- For

$$\varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta(x)\eta(y)] - \rho_0^N(x)\rho_0^N(y)$$

it holds that

$$\max_{1 \leq x < y \leq N-1} \left| \varphi_0^N(x,y) \right| \lesssim \frac{1}{N}.$$

Examples of initial measures μ_N:

- If for a given measurable profile $\rho_0 : [0,1] \to [0,1]$, we take μ_N as the Bernoulli product measure with marginal given by
$$\mu_N\{\eta : \eta(x) = 1\} = \rho_0(\tfrac{x}{N})$$
then all the conditions above are true.

- If μ_{ss} is the stationary measure, then all the conditions above are true, by choosing the profile ρ_0 as the stationary profile $\bar{\rho}$ given by

$$\bar{\rho}(q) = \begin{cases} (\beta-\alpha)q+\alpha\,;\ \theta<1, \\ \frac{\kappa(\beta-\alpha)}{2+\kappa}q+\alpha+\frac{\beta-\alpha}{2+\kappa}\,;\ \theta=1, \\ \frac{\beta+\alpha}{2}\,;\ \theta>1. \end{cases}$$

Theorem [Ornstein-Uhlenbeck limit]:

If $\{\mathscr{Y}_0^N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a mean-zero Gaussian field with covariance $\mathbb{E}\left[\mathscr{Y}_0(H)\mathscr{Y}_0(G)\right] := \sigma(H,G)$, then, the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, to a generalized Ornstein-Uhlenbeck process:

$$\partial_t \mathscr{Y}_t = \Delta_1 \mathscr{Y}_t dt + \sqrt{2\chi(\rho_t)}\nabla_1 \mathscr{W}_t,$$

where $\mathscr{W}_t$ is a space-time white noise of unit variance. As a consequence, for $H, G \in \mathcal{S}_\theta$ it holds

$$\mathbb{E}\left[\mathscr{Y}_t(H)\mathscr{Y}_s(G)\right] = \sigma(T_t^1 H, T_s^1 G) + \int_0^s \langle \nabla_1 T_{t-r}^1 H, \nabla_1 T_{s-r}^1 G\rangle_{L^{2,1}(\rho_r)} dr.$$

Above $T_t^\theta : \mathcal{S}_\theta \to \mathcal{S}_\theta$ is the semigroup associated to the PDE with the corresponding boundary conditions with $\alpha = \beta = 0$.

Associated martingales:

Let $H : [0,1] \to \mathbb{R}$ be a test function and note that

$$\mathscr{M}_t^N(H) := \mathscr{Y}_t^N(H) - \mathscr{Y}_0^N(H) - \int_0^t N^2 \mathscr{L}_N \mathscr{Y}_s^N(H)\, ds$$

is a martingale where

$$\begin{aligned} N^2 \mathscr{L}_N \mathscr{Y}_s^n(H) \;=\; & \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H(\tfrac{x}{N}) \Big(\eta_{sN^2}(x) - \rho_s^N(x) \Big) \\ & + \sqrt{N} \Big[\nabla_N^+ H(0) - H(\tfrac{1}{N}) \Big] \bar{\eta}_{sN^2}(1) \\ & + \sqrt{N} \Big[H(\tfrac{N-1}{N}) + \nabla_N^- H(1) \Big] \bar{\eta}_{sN^2}(N-1). \end{aligned}$$

Recall that for $\theta = 1$: $H'(0) = H(0)$ and $H'(1) = -H(1)$. Note that the first term at the right-hand side of the previous expression is $\mathscr{Y}_s^N(\Delta_N H)$. Above,

$$\nabla_N^+ H(x) \;=\; N \Big[H(\tfrac{x+1}{N}) - H(\tfrac{x}{N}) \Big], \;\; \nabla_N^- H(x) \;=\; N \Big[H(\tfrac{x}{N}) - H(\tfrac{x-1}{N}) \Big].$$

The correlation estimate:

Definition (Two-point correlation function)

For $x, y \in V_N = \{(x,y)\,;\, x,y \in \mathbb{N},\ 0 < x < y < N\}$, $t \in [0,T]$,

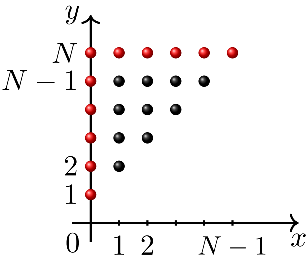
$$\varphi_t^N(x,y) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)\eta_{tN^2}(y)] - \rho_t^N(x)\rho_t^N(y)\,,$$

and set $\varphi_t^N(x,y) = 0$, for $x = 0$ or $y = N$.

Proposition:

If $\max_{x,y\in V_N} \left|\varphi_0^N(x,y)\right| \lesssim \frac{1}{N}$, then

$$\sup_{t\geq 0} \max_{(x,y)\in V_N} |\varphi_t^N(x,y)| \lesssim \frac{1}{N}.$$



$\mathscr{F}luctuations$: $\theta \neq 1$

$\mathcal{N}$on-equilibrium fluctuations ($\theta \neq 1$):

$$
\begin{aligned}
N^2 \mathcal{L}_N \mathcal{Y}_s^N(H) = & \frac{1}{\sqrt{N}} \sum_{x=1}^{N-1} \Delta_N H\Big(\tfrac{x}{N}\Big)\Big(\eta_{sN^2}(x) - \rho_s^N(x)\Big) \\
& + \sqrt{N}\nabla_N^+ H(0)\bar{\eta}_{sN^2}(1) - \sqrt{N}\nabla_N^- H(1)\bar{\eta}_{sN^2}(N-1) \\
& - \frac{N^{3/2}}{N^\theta} H\Big(\tfrac{1}{N}\Big)\bar{\eta}_{sN^2}(1) - \frac{N^{3/2}}{N^\theta} H\Big(\tfrac{N-1}{N}\Big)\bar{\eta}_{sN^2}(N-1).
\end{aligned}
$$

Lemma:

For $x \in \{1, N-1\}$ and $t \in [0,T]$ it holds that

$$
\mathbb{E}_{\mu_N}\Big[\Big(\int_0^t C_N^\theta(\eta_{sN^2}(x) - \rho_s^N(x))\, ds\Big)^2\Big] \lesssim (C_N^\theta)^2 \tfrac{N^\theta}{N^2}.
$$

We apply last result with $C_N^\theta = \sqrt{N}\mathbf{1}_{\{\theta<1\}} + N^{3/2-\theta}\mathbf{1}_{\{\theta>1\}}$.

The initial measures:

Fix $\rho_0 : [0,1] \to [0,1]$ measurable and of class C^6, and assume

$$\max_{x\in\Lambda_N} |\rho_0^N(x) - \rho_0(\tfrac{x}{N})| \lesssim \tfrac{1}{N}, \qquad \max_{(x,y)\in V_N} |\varphi_0^N(x,y)| \lesssim \tfrac{1}{N}, \tag{1}$$

$$\max_{y\in\Lambda_N} |\varphi_0^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, \ \theta \le 1, \\ \frac{1}{N}, \ \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1. \tag{2}$$

Proposition:

If (1) and (2) hold, then

$$\sup_{t\ge 0} \max_{(x,y)\in V_N} |\varphi_t^N(x,y)| \lesssim \tfrac{1}{N},$$

$$\sup_{t\ge 0} \max_{y\in\Lambda_N} |\varphi_t^N(x,y)| \lesssim \begin{cases} \frac{N^\theta}{N^2}, \ \theta \le 1, \\ \frac{1}{N}, \ \theta \ge 1, \end{cases} \qquad \text{for } x = 1, N-1.$$

Ingredients for the correlation estimate:

Show that $\rho_t^N(\cdot)$ is a solution of

$$\begin{cases} \partial_t \rho_t^N(x) \;=\; (N^2 \mathfrak{B}_N^\theta \rho_t^N)(x)\,, \quad x \in \Lambda_N\,, \quad t \geq 0\,, \\ \rho_t^N(0) \;= \alpha\,, \rho_t^N(N) \;=\; \beta\,, \quad t \geq 0\,, \end{cases}$$

where $\mathfrak{B}_N^\theta$ acts on $f : \Lambda_N \cup \{0, N\} \to \mathbb{R}$ as

$$(\mathfrak{B}_N^\theta f)(x) \;=\; \sum_{y=0}^{N} \xi_{x,y}^{N,\theta}(f(y) - f(x)), \quad \text{for } x \in \Lambda_N$$

and it is the infinitesimal generator of the RW in $\overline{\Lambda}_N$ which is absorbed at the points $\{0, N\}$. Above

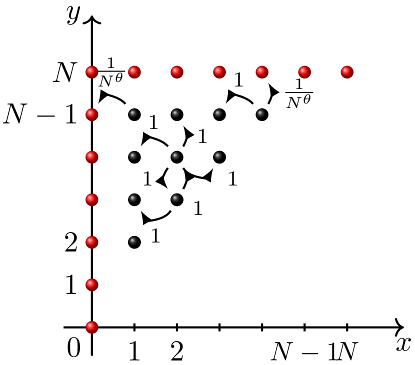
$$\xi_{x,y}^{N,\theta} \;=\; \begin{cases} 1\,, & \text{if } \;|y - x| = 1 \text{ and } x, y \in \Lambda_N, \\ N^{-\theta}\,, & \text{if } \; x = 1, y = 0 \text{ and } x = N-1\,, y = N, \\ 0\,, & \text{otherwise.} \end{cases}$$

Ingredients for the correlation estimate:

Then $\varphi_t^N(x,y)$ is solution of

$$\begin{cases} \partial_t \varphi_t^N(x,y) = N^2 \mathcal{A}_N^\theta \varphi_t^N(x,y) - (\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1}, (x,y) \in V_N, \\ \varphi_t^N(x,y) = 0, (x,y) \in \partial V_N. \end{cases}$$

Above $\mathcal{A}_N^\theta$ is the infinitesimal generator of a random walk $\mathcal{X}_{tN^2}$ in $V_N \cup \partial V_N$ which is absorbed at ∂V_N with rate $N^{-\theta}$.



How to get bounds?

By Duhamel's formula

$$\varphi_t^N(x,y) = \mathbb{E}_{(x,y)}\Big[\varphi_0^N(\mathcal{X}_{tN^2}) + \int_0^t g_{t-s}^N(\mathcal{X}_{sN^2})\mathbf{1}(\mathcal{X}_{sN^2} \in \mathcal{D}_N^+)\, ds\Big],$$

where $\mathbb{E}_{(x,y)}$ denotes the expectation of the law of the walk $\{\mathcal{X}_{tN^2}; t \geq 0\}$ starting from (x,y),
$g_t^N(x,x+1) = -\left(\nabla_N^+ \rho_t^N(x)\right)^2$, and
$\mathcal{D}_N^+ := \{(x,y) \in V_N : \quad y = x+1\}$. Therefore, it is enough to estimate

$$\max_{\substack{(z,w)\in V_N \\ z\neq w}} |\varphi_0^N(z,w)| + \sup_{t\geq 0} \max_{z\in\Lambda_{N-1}} |g_t^N(z,z+1)| \max_{\substack{(x,y)\in V_N \\ x\neq y}} T_N(x,y),$$

where

$$T_N(x,y) := \mathbb{E}_{(x,y)}\left[\int_0^\infty \mathbf{1}(\mathcal{X}_{tN^2} \in \mathcal{D}_N^+)dt\right]$$

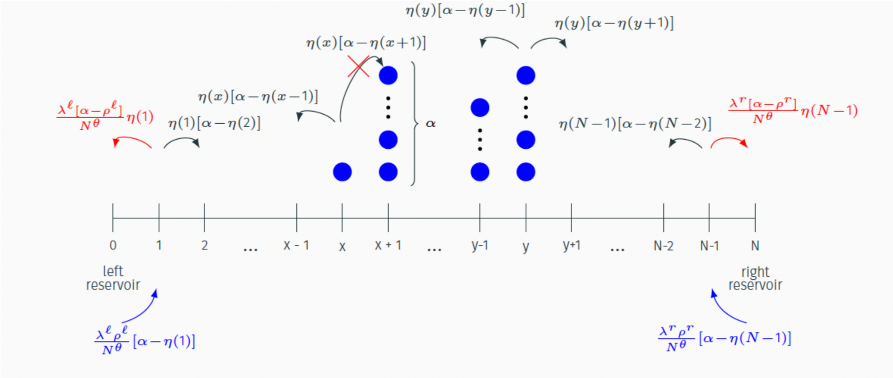
is the time spent by the walk on the diagonal $\mathcal{D}_N^+$.

Estimating the correlation function?

For estimating g_t^N we need to estimate the time spent by a $1-d$ RW at the points $x=1$ and $x=N-1$ and if it is the absorbed one, then this is of order $O(\frac{N^\theta}{N^2})$ (good bound when $\theta<1$ but not when $\theta>1$). When $\theta>1$ we compare with the reflected RW and we prove that the time now is of order $O(\frac{1}{N})$.

For estimating φ_t^N we also need the same type of estimates in the $2-d$ setting for the time spent by the RW on the diagonal $\mathcal{D}_N^+$. For $\theta<1$ we use the absorbed RW but for $\theta>1$ we use the reflected RW.

Partial Exclusion:



$\alpha \in \mathbb{N}$, $\lambda_\ell, \lambda_r \in (0,1]$ and $\rho_\ell, \rho_r \in (0,\alpha)$, $\theta \in \mathbb{R}$.

$\mathscr{H}ydrodynamic\ \mathscr{L}imit$:

Theorem [Franceschini, G., Salvador]:

Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to $g(\cdot)$. Then, for any $0 \leq t \leq T$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\Big|\frac{1}{N}\sum_{x\in\Lambda_N} H(\tfrac{x}{N})\eta_{tN^2}(x) - \int_0^1 H(q)\rho(t,q)dq\Big| > \delta\Big) = 0$$

and $\rho_t(\cdot)$ is the UNIQUE weak solution of the heat equation $\partial_t \rho_t(q) = \alpha \partial_q^2 \rho_t(q)$ with

- $\theta > 1$ **Neumann b.c.:** $\partial_q \rho_t(0) = \partial_q \rho_t(1) = 0$.
- $\theta = 1$ **Robin b.c.:**
  $\partial_q \rho_t(0) = \lambda^\ell(\rho_t(0) - \rho^\ell) \qquad \partial_q \rho_t(1) = \lambda^r(\rho^r - \rho_t(1))$.
- $\theta < 1$ **Dirichlet b.c.:** $\rho_t(0) = \rho^\ell$, $\rho_t(1) = \rho^r$.

Different definition?

It would be natural to extend φ_t^N to the diagonal by

$$\varphi_t^N(x,x) := \mathbb{E}_{\mu^N}[(\eta_{tN^2}(x) - \rho_t^N(x))^2].$$

A more convenient definition is to extend it as

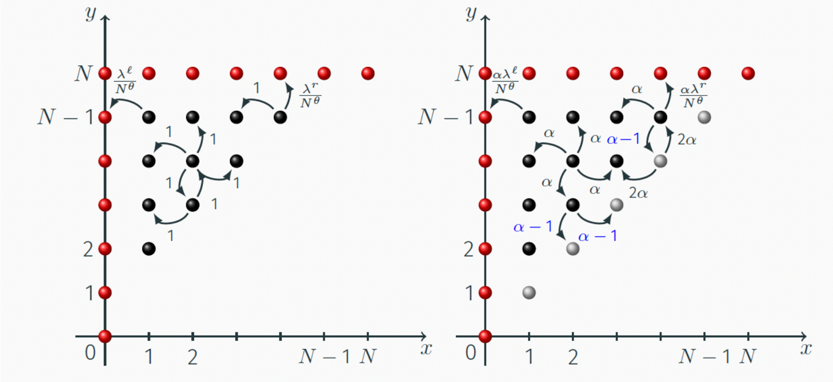
$$\varphi_t^N(x,x) := \mathbb{E}_{\mu^N}\Big[\frac{\alpha}{\alpha-1}\eta_{tN^2}(x)(\eta_{tN^2}(x)-1) - \rho_t^N(x)^2\Big].$$

In this case we also have

$$\begin{cases} \partial_t \varphi_t^N(x,y) = N^2 \mathcal{A}_N^\theta \varphi_t^N(x,y) - (\nabla_N^+ \rho_t^N(x))^2 \delta_{y=x+1}, (x,y) \in V_N, \\ \varphi_t^N(x,y) = 0, (x,y) \in \partial V_N, \\ \varphi_0^N(x,y) = \mathbb{E}_{\mu_N}[\eta_0(x)\eta_0(y)] - \rho_0^N(x)\rho_0^N(y), (x,y) \in V_N \cup \partial V_N, \end{cases}$$

but the operator $\mathcal{A}_N^\theta$ is different from the one in SEP(1).

Another RW



(I JUMP TO THE BOARD NOW!).

What about θ negative?

We expect the same behavior as $\theta \in [0,1)$ - DIRICHLET!
In this case the space of test functions that we take is $\mathcal{S}_\theta$ composed of functions $H \in C^\infty([0,1])$ such that for any $k \in \mathbb{N} \cup \{0\}$ it holds $\partial_u^k H(0) = \partial_u^k H(1) = 0$.
Now all the derivatives are null at the boundary!!! Our result says that if $\theta < 0$, the sequence $\{\mathbb{Q}_N\}_{N \in \mathbb{N}}$ converges, as $N \to +\infty$, to the generalized Ornstein-Uhlenbeck process as in the case $\theta \in [0,1)$ since $\mathcal{Y}_t$ satisfies the conditions:

1. regularity condition: $\mathbb{E}[(\mathcal{Y}_t(H))^2] \lesssim \|H\|_{L^2}$, for any $H \in \mathcal{S}_\theta$;
2. boundary condition: for any $t \in [0,T]$ and $j \in \{0,1\}$, it holds that

$$\lim_{\epsilon \to 0} \mathbb{E}_{\mu_N}\left[\left(\int_0^t \mathcal{Y}_s(\iota_\epsilon^j)ds\right)^2\right] = 0,$$

where for $u \in [0,1]$ and: $j = 0$, $\iota_\epsilon^0(u) := \epsilon^{-1}\mathbf{1}_{(0,\epsilon]}(u)$ and, for $j = 1$, $\iota_\epsilon^1(u) := \epsilon^{-1}\mathbf{1}_{[1-\epsilon,1)}(u)$.

And now?

1. Estimates for k-points correlations?
   1. How to define the correlation function?
   2. How to control static equations?
   3. Is the bound equal to SEP(1)?
2. What about other models with duality?

Thank you!
Merci!
Obrigada!