Duality for a boundary driven asymmetric model of energy transport

Chiara Franceschini

University of Modena and Reggio Emilia

Joint work with Gioia Carinci and Francesco Casini

Inhomogeneous Random Systems

Paris, January 24, 2024

Motivation: to have a particle system with an *asymmetric bulk* but also an *open boundary* with duality property.

Motivation: to have a particle system with an *asymmetric bulk* but also an *open boundary* with duality property.

Idea: start with a *symmetric bulk* and a *close boundary* with duality property.

Motivation: to have a particle system with an *asymmetric bulk* but also an *open boundary* with duality property.

Idea: start with a *symmetric bulk* and a *close boundary* with duality property.

- bulk: one dimensional finite chain $\Lambda_N = \{1, \dots, N-1\}$
- symmetric bulk: no preference to go left or right (nearest neighbors)
- close boundary: no injection nor absorption of particles (conservation)

This idea goes back to Kipnis-Marchioro-Presutti in 1982.

KMP 1982

$L^{KMP} = L_{left} + \sum_{i\in\Lambda_N} L_{i,i+1} + L_{right}$ where

$$L_{i,i+1}f(\mathbf{z}) = \int_0^1 dp\Big[f(z_1,\dots,p(z_i+z_{i+1}),(1-p)(z_i+z_{i+1}),\dots z_N) - f(\mathbf{z})\Big]$$

$$L_{left}f(\mathbf{z}) = \int_0^\infty dz_1' \frac{e^{-z_1'/T_-}}{T_-}\left[f(z_1',\dots,z_N) - f(\mathbf{z})\right]$$

$$L_{right}f(\mathbf{z}) = \int_0^\infty dz_N' \frac{e^{-z_1'/T_+}}{T_+}\left[f(z_1,\dots,z_N') - f(\mathbf{z})\right]$$

for $z \in \mathbb{R}_+^{\Lambda_N}$, z_i represents the energy at site i

After the redistribution: with $p \sim U([0,1])$
$z_i' = p(z_i + z_{i+1})$ is the new energy at site i and
$z_{i+1}' = (1-p)(z_i+z_{i+1})$ is the new energy at site $i+1$

KMP 1982

$L^{KMP} = L_{left} + \sum_{i \in \Lambda_N} L_{i,i+1} + L_{right}$ where

$$L_{i,i+1} f(\mathbf{z}) = \int_0^1 dp \Big[f(z_1, \dots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \dots z_N) - f(\mathbf{z}) \Big]$$

$$L_{left} f(\mathbf{z}) = \int_0^\infty dz_1' \frac{e^{-z_1'/T_-}}{T_-} \left[f(z_1', \dots, z_N) - f(\mathbf{z}) \right]$$

$$L_{right} f(\mathbf{z}) = \int_0^\infty dz_N' \frac{e^{-z_1'/T_+}}{T_+} \left[f(z_1, \dots, z_N') - f(\mathbf{z}) \right]$$

for $z \in \mathbb{R}_+^{\Lambda_N}$, z_i represents the energy at site i

After the redistribution: with $p \sim U([0,1])$
$z_i' = p(z_i + z_{i+1})$ is the new energy at site i and
$z_{i+1}' = (1-p)(z_i + z_{i+1})$ is the new energy at site $i+1$

They proved the Fourier's law: $Q = -\frac{dT(u)}{dt}, \quad u \in (0,1)$

Duality property

Definition

Let $(z_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ two Markov processes on Ω and Ω^{dual} with generators L and L^{dual}, respectively. z_t is **dual** to ξ_t with duality function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ if $\forall t \geq 0$,

$$\mathbb{E}_z(D(z_t, \xi)) = \mathbb{E}_\xi(D(z, \xi_t)) \qquad \forall (z, \xi) \in \Omega \times \Omega^{dual} .$$

Equivalently,

$$LD(\cdot, \xi)(z) = L^{dual} D(z, \cdot)(\xi)$$

Duality enables to connect, via a duality function, the process of interest to another one which is a **simpler** process.

Symmetric Brownian Energy Process

The symmetric BEP is an interacting diffusion process which describes the symmetric energy exchange between nearest neighboring sites and it conserves the total energy of the chain. Let z_i represent the energy at site $i \in \Lambda_N$, $\boldsymbol{z} \in \mathbb{R}_+^{\Lambda_N}$, and $s > 0$.

$$L^{\mathrm{BEP}} = \sum_{i=1}^{N-1} L_{i,i+1} \quad \text{where the local generator is}$$

$$L_{i,i+1} = z_i z_{i+1} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2 - 2s\,(z_i - z_{i+1}) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)$$

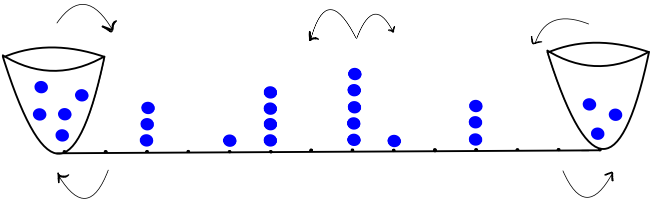
REMARK: The symmetric BEP is obtained as a rescaling of the Symmetric Inclusion Process (SIP)

Adding an open boundary

Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs whose role is to destroy the conservation of particles/energy and create a flux into the bulk:

$$L = L_{left} + L_{bulk} + L_{right}$$

L_{left} and L_{right} are chosen such that in equilibrium there still is reversibility.



But in general, the processes now are no longer reversible.

Open BEP [Giardinà, Redig, Kurchan 2008]

The generator is $L^{\mathrm{BEP}} = L_{left} + \sum_{i=1}^{N} L_{i,i+1} + L_{right}$ where

$$L_{left} = T_- \left(2s\partial_{z_1} + z_1\partial_{z_1}^2\right) - \frac{1}{2} z_1 \partial_{z_1}$$

$$L_{right} = T_+ \left(2s\partial_{z_N} + z_N\partial_{z_N}^2\right) - \frac{1}{2} z_N \partial_{z_N}$$

- **Invariant measure in equilibrium ($T_- = T_+ = T$):** Homogeneous product of Gamma distribution with shape parameter $2s > 0$ and scale parameter $T > 0$

$$\mu_{N,T}(z) = \prod_{i=1}^{N} \frac{z_i^{2s-1}}{\Gamma(2s)} \frac{e^{-z_i/T}}{T^{2s}}$$

Open BEP [Giardinà, Redig, Kurchan 2008]

The generator is $L^{\mathrm{BEP}} = L_{left} + \sum_{i=1}^{N} L_{i,i+1} + L_{right}$ where

$$L_{left} = T_{-}\left(2s\partial_{z_1} + z_1\partial_{z_1}^2\right) - \frac{1}{2}z_1\partial_{z_1}$$

$$L_{right} = T_{+}\left(2s\partial_{z_N} + z_N\partial_{z_N}^2\right) - \frac{1}{2}z_N\partial_{z_N}$$

- **Invariant measure in equilibrium ($T_{-} = T_{+} = T$):**
  Homogeneous product of Gamma distribution with shape parameter $2s > 0$ and scale parameter $T > 0$

$$\mu_{N,T}(z) = \prod_{i=1}^{N} \frac{z_i^{2s-1}}{\Gamma(2s)} \frac{e^{-z_i/T}}{T^{2s}}$$

- **Invariant measure in non-equilibrium ($T_{-} \neq T_{+}$):** $\mu_{N,T_{-},T_{+}}$ **?**

Thermalization limit: from BEP($2s$) to KMP

Thermalization procedure: consider the bond $(i, i+1)$ and then redistribute the energies on that bond, z_i and z_{i+1} according to the stationary measure of the bond, conditioning to the conservation of the total energy of the bond. For the BEP z_i and z_{i+1} are independent Gamma $(2s, T)$, then

$$\frac{z_x}{z_x + z_{x+1}} | z_x + z_{x+1} = E \sim Beta(2s, 2s)$$

Then for $s = 1/2$ you get the uniform distribution $[0, 1]$ for the bulk.

For the boundary, the stationary distribution of the Brownian Energy reservoir is also Gamma $(2s, T)$ then for $s = 1/2$ one gets the exponential distribution with parameter $1/T$.

Asymmetric Brownian Energy Process: x_t

Introduced without boundary in 2016 by Carinci, Giardinà, Redig and Sasamoto.

$$\mathcal{L}^{\mathrm{ABEP}} = \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{\mathrm{ABEP}}$$

where, for $i \in \Lambda_N$, the action on smooth functions $f : \mathbb{R}_+^{\Lambda_N} \to \mathbb{R}$ is

Asymmetric Brownian Energy Process: x_t

Introduced without boundary in 2016 by Carinci, Giardinà, Redig and Sasamoto.

$$\mathcal{L}^{\mathrm{ABEP}} = \sum_{i=1}^{N-1} \mathcal{L}^{\mathrm{ABEP}}_{i,i+1}$$

where, for $i \in \Lambda_N$, the action on smooth functions $f : \mathbb{R}_+^{\Lambda_N} \to \mathbb{R}$ is

$$\begin{aligned}\left[\mathcal{L}^{\mathrm{ABEP}}_{i,i+1} f\right](\boldsymbol{x}) &= \frac{1}{2\sigma^2}\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 f(\boldsymbol{x}) \\ &+ \frac{1}{\sigma}\left(\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right) + 2s\left(2 - e^{-\sigma x_i} - e^{\sigma x_{i+1}}\right)\right) \cdot \\ &\cdot \left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) f(\boldsymbol{x})\end{aligned}$$

Asymmetric Brownian Energy Process: x_t

Introduced without boundary in 2016 by Carinci, Giardinà, Redig and Sasamoto.

$$\mathcal{L}^{\mathrm{ABEP}} = \sum_{i=1}^{N-1} \mathcal{L}_{i,i+1}^{\mathrm{ABEP}}$$

where, for $i \in \Lambda_N$, the action on smooth functions $f : \mathbb{R}_+^{\Lambda_N} \to \mathbb{R}$ is

$$\begin{aligned}\left[\mathcal{L}_{i,i+1}^{\mathrm{ABEP}} f\right](\boldsymbol{x}) &= \frac{1}{2\sigma^2}\left(1-e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}}-1\right)\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right)^2 f(\boldsymbol{x})\\ &+\frac{1}{\sigma}\left(\left(1-e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}}-1\right)+2s\left(2-e^{-\sigma x_i}-e^{\sigma x_{i+1}}\right)\right)\cdot\\ &\cdot\left(\frac{\partial}{\partial x_{i+1}}-\frac{\partial}{\partial x_i}\right) f(\boldsymbol{x})\end{aligned}$$

REMARK: for $\sigma \to 0$ then *ABEP* $\to$ *BEP*.

$BEP \to ABEP$?

$BEP \to ABEP$?

Definition

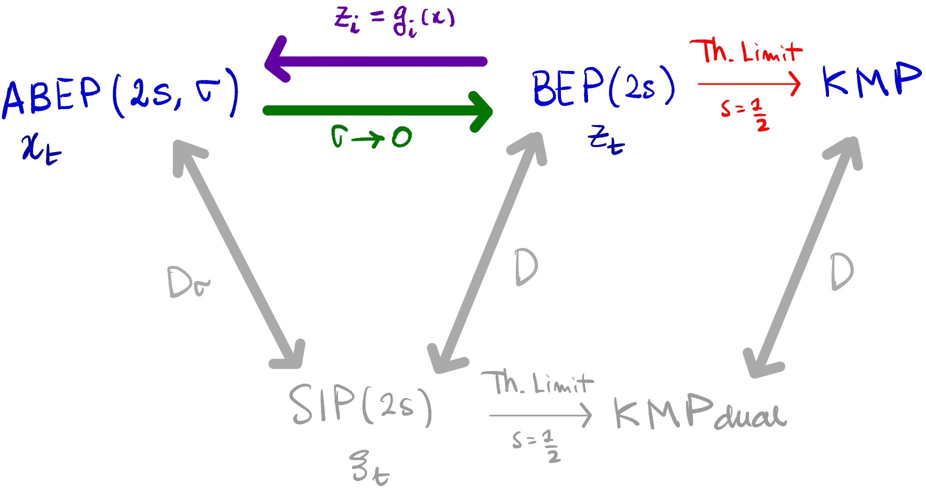
We define the map $g : \mathbb{R}_+^{\Lambda_N} \to \mathbb{R}_+^{\Lambda_N}$ via

$$g(x) = (g_i(x))_{i \in \Lambda_N} \qquad \text{with} \qquad g_i(x) := \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma}$$

where $E_i(x)$ denotes the energy of the system at the right of site $i \in \Lambda_N$, i.e.

$$E_i(x) = \sum_{\ell=i}^{N} x_\ell \quad \text{for} \quad i = 1, \ldots, N \qquad \text{with} \quad E_{N+1}(x) = 0 \, .$$

Thanks to this map, we have several results.


$z_i = g_i(x)$
ABEP(2s, σ)
x_t
σ → 0
BEP(2s)
z_t
Th. Limit
s = 1/2
KMP
Dσ
D
D
SIP(2s)
ξ_t
Th. Limit
s = 1/2
KMPdual

Results via the non-local map g

The ABEP can be found for all $i \in \Lambda_N$ as

$$\left[L_{i,i+1}^{\mathrm{BEP}} f\right](g(x)) = \left[\mathcal{L}_{i,i+1}^{\mathrm{ABEP}} f \circ g\right](x)$$

This implies the following nice properties

- **Reversibility for ABEP:**

$$\begin{aligned}\mu_{N,T}^{\mathrm{ABEP}}(x) &= \mu_{N,T}^{\mathrm{BEP}}(g(x)) \\ &= \exp\left\{\frac{e^{-\sigma E(x)} - 1}{\sigma T}\right\} \prod_{i=1}^{N} \frac{\left(1 - e^{-\sigma x_i}\right)^{(2s-1)}}{\Gamma(2s)\sigma^{2s-1} T^{2s}} \; e^{-\sigma x_i (2s(i-1)+1)}\end{aligned}$$

- **Duality for ABEP:**

$$D_{\sigma}^{\mathrm{ABEP}}(x, \xi) = D^{\mathrm{BEP}}\left((g(x)), \xi\right)$$

where ξ_t is the Symmetric Inclusion Process with absorbing boundary.

Symmetric Inclusion Process: bulk

The SIP describes particles jumping to nearest neighbor sites with an inclusion interaction. Let ξ_i represent the number of particles at site $i \in \Lambda_N$, $\boldsymbol{\xi} \in \mathbb{N}^{\Lambda_N}$, and $s > 0$.

$$L^{\mathrm{SIP}} = \sum_{i=1}^{N-1} L_{i,i+1} \quad \text{where the action on functions is}$$

$$\begin{aligned} L_{i,i+1} f(\boldsymbol{\xi}) &= \xi_i (2s + \xi_{i+1}) \left[f(\boldsymbol{\xi}^{i,i+1}) - f(\boldsymbol{\xi}) \right] \\ &+ \xi_{i+1} (2s + \xi_i) \left[f(\boldsymbol{\xi}^{i+1,1}) - f(\boldsymbol{\xi}) \right] \end{aligned}$$

Above $\boldsymbol{\xi}^{i,j} := \boldsymbol{\xi} - \boldsymbol{\delta}_i + \boldsymbol{\delta}_j$

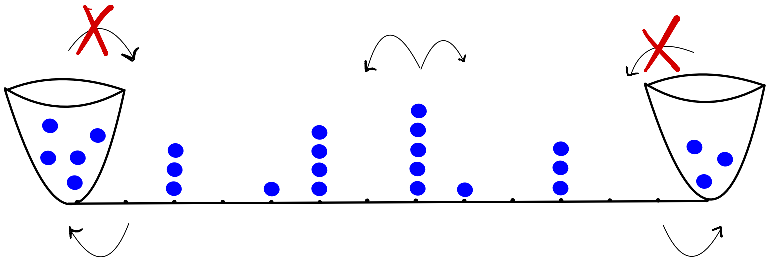
Dual SIP

The generator is $L^{dual} = L_{left} + \sum_{i \in \Lambda_N} L_{i,i+1} + L_{right}$ where

$$L_{i,i+1} = \text{is the bulk of the SIP}$$

$$L_{left} = \xi_1 \left[f(\boldsymbol{\xi}^{1,0}) - f(\boldsymbol{\xi}) \right]$$

$$L_{right} = \xi_N \left[f(\boldsymbol{\xi}^{N,N+1}) - f(\boldsymbol{\xi}) \right]$$



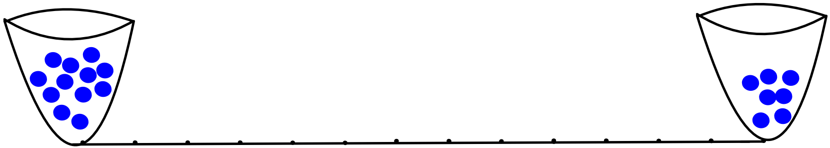
Dual SIP

The generator is $L^{dual} = L_{left} + \sum_{i\in\Lambda_N} L_{i,i+1} + L_{right}$ where

$$L_{i,i+1} = \text{is the bulk of the SIP}$$

$$L_{left} = \xi_1 \left[f(\boldsymbol{\xi}^{1,0}) - f(\boldsymbol{\xi}) \right]$$

$$L_{right} = \xi_N \left[f(\boldsymbol{\xi}^{N,N+1}) - f(\boldsymbol{\xi}) \right]$$



- the dual process is only absorbing.
- the dual process conserves the total number of particles: $\boldsymbol{\xi} \in \mathbb{N}^{0\cup\Lambda_N\cup N+1}$

Duality results 1

$$\text{Let } D(z,\xi) = \prod_{i=1}^{N} d(z_i,\xi_i) \text{ then}$$

Theorem

L^{BEP} and L^{SIP} are dual with:

$$d^{cl}(z,\xi) = \frac{z^{\xi}}{\Gamma(\xi+2s)} \qquad \textit{and} \qquad d^{or}(z,\xi) = (-T)^{\xi}\ {}_1F_1\left(\begin{matrix}-\xi\\ 2s\end{matrix}\,\middle|\,\frac{z}{T}\right)$$

Above ${}_1F_1\left(\begin{smallmatrix}-n\\ \alpha\end{smallmatrix}\middle|\, y\right)$ is a Laguerre polynomial in y of degree n and $\alpha > 0$.

These two results were proved in many past works via different techniques by several people: Carinci, Floreani, F., Giardinà, Kurchan, Redig, Sau, Vafayi.

Duality results 2

Theorem

Open BEP and absorbing SIP are dual with:

$$D^{cl}(z,\xi) = (T_-)^{\xi_0} \prod_{i=1}^{N} d^{cl}(z_i,\xi_i)\,(T_+)^{\xi_{N+1}} \qquad \text{and}$$

$$D^{or}(z,\xi) = (T_- - T)^{\xi_0} \prod_{i=1}^{N} d^{or}(z_i,\xi_i) \quad (T_+ - T)^{\xi_{N+1}}$$

Theorem (Duality result 3, Carinci, Giardinà, Redig and Sasamoto)

$$\mathcal{L}^{\text{ABEP}} D_\sigma^{\text{ABEP}}(\cdot,\xi)(x) = L^{\text{SIP}} D_\sigma^{\text{ABEP}}(x,\cdot)(\xi)$$

where the classical duality function, in particular, reads:

$$D_\sigma^{\text{ABEP}}(x,\xi) = \prod_{i=1}^{N} \frac{1}{\Gamma(2s+\xi_i)} \left(\frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma} \right)^{\xi_i}$$

The dual process is the same: the <u>symmetric</u> inclusion process!

REMARK: The dependence on the asymmetry parameter $\sigma > 0$ only appears in $\mathcal{L}^{\text{ABEP}}$ and D_σ^{ABEP}.

Theorem (Duality result 3, Carinci, Giardinà, Redig and Sasamoto)

$$\mathcal{L}^{\text{ABEP}} D_\sigma^{\text{ABEP}}(\cdot, \xi)(x) = L^{\text{SIP}} D_\sigma^{\text{ABEP}}(x, \cdot)(\xi)$$

where the classical duality function, in particular, reads:

$$D_\sigma^{\text{ABEP}}(x, \xi) = \prod_{i=1}^{N} \frac{1}{\Gamma(2s + \xi_i)} \left(\frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma} \right)^{\xi_i}$$

The dual process is the same: the symmetric inclusion process!

REMARK: The dependence on the asymmetry parameter $\sigma > 0$ only appears in $\mathcal{L}^{\text{ABEP}}$ and D_σ^{ABEP}.

Can we generalize this even further by adding two reservoirs at sites $i = 0$ and $i = N + 1$?

Open ABEP

$$\mathcal{L}^{\mathrm{ABEP}} = \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{left}} + \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{bulk}} + \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{right}}$$

where

$$[\mathcal{L}^{\mathrm{ABEP}}_{\mathrm{left}} f](\boldsymbol{x}) = T_- \left(e^{\sigma E(x)} \left(2s - 1 + e^{\sigma x_1}\right) \frac{\partial}{\partial x_1} + \frac{e^{\sigma E(x)}}{\sigma} \left(e^{\sigma x_1} - 1\right) \frac{\partial^2}{\partial x_1^2} \right) f(\boldsymbol{x}) - \frac{e^{\sigma x_1} - 1}{\sigma} \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \qquad \text{and}$$

Open ABEP

$$\mathcal{L}^{\mathrm{ABEP}} = \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{left}} + \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{bulk}} + \mathcal{L}^{\mathrm{ABEP}}_{\mathrm{right}}$$

where

$$[\mathcal{L}^{\mathrm{ABEP}}_{\mathrm{left}} f](\boldsymbol{x}) = T_{-} \left(e^{\sigma E(x)} \left(2s - 1 + e^{\sigma x_1}\right) \frac{\partial}{\partial x_1} + \frac{e^{\sigma E(x)}}{\sigma} \left(e^{\sigma x_1} - 1\right) \frac{\partial^2}{\partial x_1^2} \right) f(\boldsymbol{x}) - \frac{e^{\sigma x_1} - 1}{\sigma} \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \qquad \text{and}$$

$$\begin{aligned}
[\mathcal{L}^{\mathrm{ABEP}}_{\mathrm{right}} f](x) &= \left(2s T_{+} - \frac{1 - e^{-\sigma x_N}}{\sigma} \right) \sum_{l=1}^{N} e^{\sigma E_l(x)} \left(\partial_{x_l} - \partial_{x_{l-1}} \right) f(\boldsymbol{x}) \\
&\quad + T_{+} \frac{1 - e^{-\sigma x_N}}{\sigma} \sum_{l,j=1}^{N} e^{\sigma (E_l(x) + E_j(x))} \left(\partial_{x_l} - \partial_{x_{l-1}} \right) \left(\partial_{x_j} - \partial_{x_{j-1}} \right) f(\boldsymbol{x}) \\
&\quad + T_{+} \left(1 - e^{-\sigma x_N} \right) \sum_{l=1}^{N} e^{2\sigma E_l(x)} \left(\partial_{x_l} - \partial_{x_{l-1}} \right) f(\boldsymbol{x})
\end{aligned}$$

Duality result 4

Theorem (Carinci, Casini, F. 2023)

The ABEP$(2s,\sigma)$ with open boundary is dual to the SIP$(2s)$ with only absorbing reservoirs with respect to the following duality function

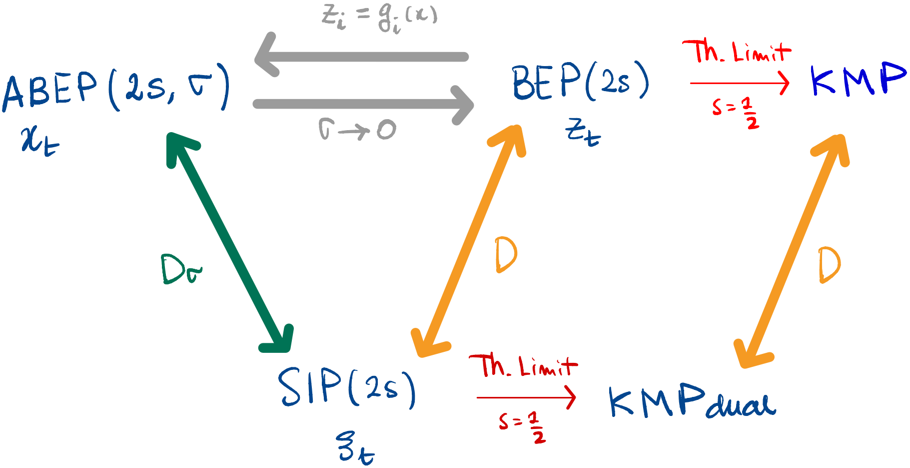
$$D_\sigma(x,\xi) = (T_-)^{\xi_0} \cdot \prod_{i=1}^{N} \frac{1}{\Gamma(2s+\xi_i)} (g_i(x))^{\xi_i} \cdot (T_+)^{\xi_{N+1}}$$

What can we do with it?

We can compute “moments” using the definition of duality,

$$\mathbb{E}_x\left[D(x_t,\xi)\right] = \mathbb{E}_\xi\left[D(x,\xi_t)\right] \text{ where we choose } \xi = \delta_i$$

Duality scheme



Exponential moments with respect to the stationary measure μ_{ss}

$$D_\sigma(x,\delta_i) = \frac{\Gamma(2s)}{\Gamma(2s+1)} g_i(x) = \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{2s\ \sigma}$$

Because of duality

$$\mathbb{E}_{\mu_{ss}} D_\sigma(x,\delta_i) = \lim_{t\to+\infty} \mathbb{E}_x D_\sigma(x_t,\delta_i) = \lim_{t\to+\infty} \mathbb{E}_{\delta_i} D_\sigma(x,\delta_{i(t)}) =$$

$$\underbrace{D_\sigma(x,\delta_0)}_{T_-} \underbrace{\mathbb{P}_{\delta_i}\left(\delta_{i(\infty)} = \delta_0\right)}_{1-\frac{i}{N+1}} + \underbrace{D_\sigma(x,\delta_{N+1})}_{T_+} \underbrace{\mathbb{P}_{\delta_i}\left(\delta_{i(\infty)} = \delta_{N+1}\right)}_{\frac{i}{N+1}}$$

Namely,

$$\mathbb{E}_{\mu_{ss}}\left[\frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{2s\ \sigma}\right] = T_- + \frac{i}{N+1}\left(T_+ - T_-\right)$$

Comparison with the symmetric case

$$D(z,\delta_i) = \frac{\Gamma(2s)}{\Gamma(2s+1)} z_i = \frac{z_i}{2s}$$

Because of duality

$$\mathbb{E}_{\mu_{ss}} D(z,\delta_i) = \lim_{t\to+\infty} \mathbb{E}_z D(z_t,\delta_i) = \lim_{t\to+\infty} \mathbb{E}_{\delta_i} D(x,\delta_{i(t)}) =$$

$$\underbrace{D(z,\delta_0)}_{T_-} \underbrace{\mathbb{P}_{\delta_i}\left(\delta_{i(\infty)} = \delta_0\right)}_{1-\frac{i}{N+1}} + \underbrace{D(z,\delta_{N+1})}_{T_+} \underbrace{\mathbb{P}_{\delta_i}\left(\delta_{i(\infty)} = \delta_{N+1}\right)}_{\frac{i}{N+1}}$$

Namely,

$$\mathbb{E}_{\mu_{ss}}\left[\frac{z_i}{2s}\right] = T_- + \frac{i}{N+1}\left(T_+ - T_-\right)$$

Cole-Hopf/Gärtner transformation

$$\mathbb{E}_{\mu_{ss}}\left[\frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{2s\ \sigma}\right] = T_- + \frac{i}{N+1}\left(T_+ - T_-\right)$$

Summing on both sides from m to N we finally get

$$\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)}\right] = 1 - 2s\ \sigma\ T_-(N-m+1) + \frac{2s\ \sigma}{N+1}\left(T_+ - T_-\right)\frac{(m+N)(m-N-1)}{2}\ .$$

We can push this up to two points correlations

Cole-Hopf/Gärtner transformation

For $i \neq j$

$$\mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\frac{g_j(x)}{2s\ \sigma}\right] - \mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\right]\mathbb{E}_{\mu_{ss}}\left[\frac{g_j(x)}{2s\ \sigma}\right] = (T_- - T_+)^2 \frac{i}{(N+1)^2}\frac{N+1-j}{2s(N+1)+1}$$

This gives information on $\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)} e^{-\sigma E_n(x)}\right]$

Cole-Hopf/Gärtner transformation

For $i \neq j$

$$\mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\frac{g_j(x)}{2s\ \sigma}\right] - \mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\right]\mathbb{E}_{\mu_{ss}}\left[\frac{g_j(x)}{2s\ \sigma}\right]$$
$$= (T_- - T_+)^2 \frac{i}{(N+1)^2}\frac{N+1-j}{2s(N+1)+1}$$

This gives information on $\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)}e^{-\sigma E_n(x)}\right]$

This suggests that this transformation allows to go from heat equation to Burger equation, similarly to the case of exclusion process.

Cole-Hopf/Gärtner transformation

For $i \neq j$

$$\mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\frac{g_j(x)}{2s\ \sigma}\right] - \mathbb{E}_{\mu_{ss}}\left[\frac{g_i(x)}{2s\ \sigma}\right]\mathbb{E}_{\mu_{ss}}\left[\frac{g_j(x)}{2s\ \sigma}\right]$$
$$= (T_- - T_+)^2 \frac{i}{(N+1)^2}\frac{N+1-j}{2s(N+1)+1}$$

This gives information on $\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)} e^{-\sigma E_n(x)}\right]$

This suggests that this transformation allows to go from heat equation to Burger equation, similarly to the case of exclusion process.

Thanks for the attention!