Effect of boundary on the large deviations of density and of current in the NESS of SSEP: microscopic and hydrodynamics

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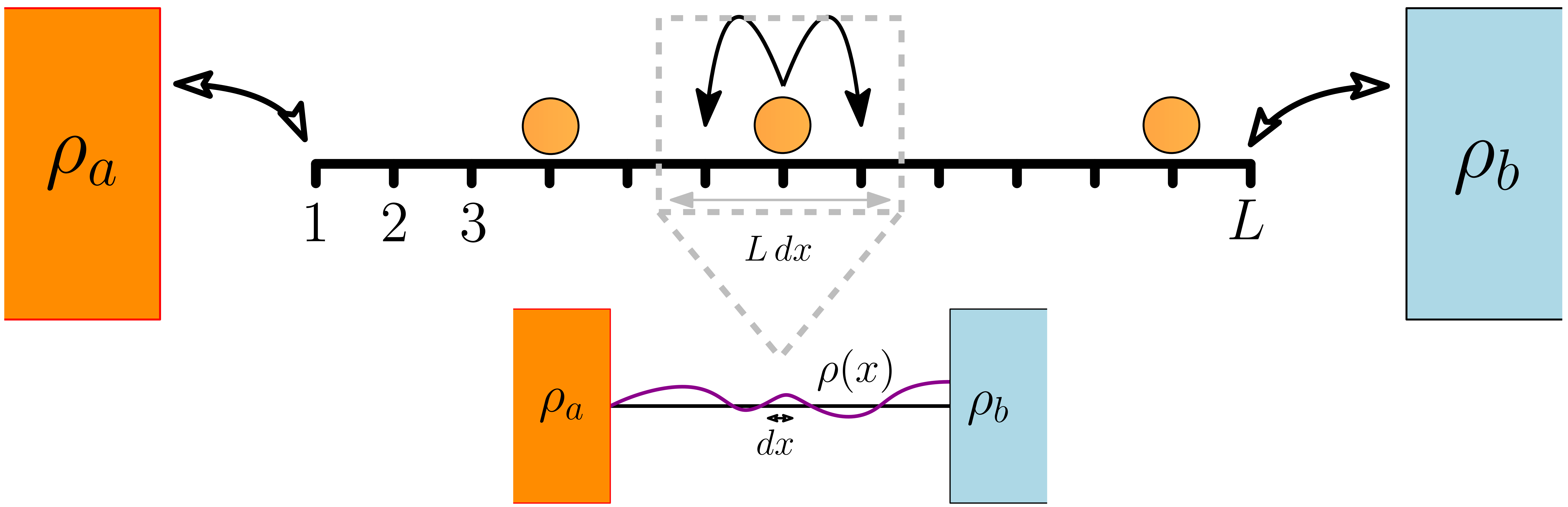
Derrida, Hirschberg, TS, J Stat Phys [2020]
Soumyabrata Saha and TS, arXiv [2023]

What is it about?



Steady state density distribution

$$P[\rho(x)] \sim e^{-L\,\mathscr{F}[\rho(x)]}$$

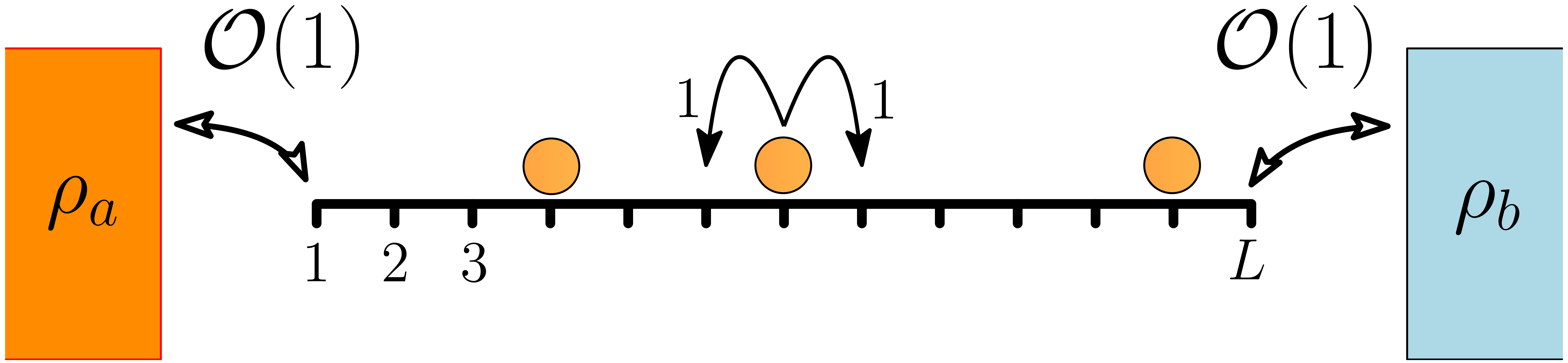
Statistics of current

$$P\Big(j = \frac{Q}{t}\Big) \sim e^{-\frac{t}{L}\,\mathscr{R}(j\,L)}$$

Characterises the Macroscopic fluctuations

Two famous results

Symmetric simple exclusion process



Random walker with hard core repulsion

Density

$$P[\rho(x)] \sim e^{-L\,\mathcal{F}[\rho(x)]}$$

Exact microscopic solution

Derrida, Lebowitz, Speer (PRL, 2000) (JSP 2002)

Hydrodynamics

Bertini, De Sole, Gabrielli, Jona-Lassinio, Landim (PRL, 2000) (JSP 2002)

Current

$$P\left(j\right) \sim e^{-\frac{T}{L}\,\mathcal{R}(j\,L)}$$

Exact microscopic solution

Bodineau, Derrida, (PRL, 2004)
Derrida, Doucot, Roche (JSP, 2004)

Hydrodynamics

Bertini, De Sole, Gabrielli, Jona-Lassinio, Landim (PRL, 2005) (JSP 2006)

Macroscopic fluctuation theory

Bertini et al Rev Mod Phys (2016)

Density LDF

$$\mathscr{F}[\rho(x)] = \int_0^1 dx \left\{ \rho(x) \ln \frac{\rho(x)}{F(x)} + (1-\rho(x)) \ln \frac{1-\rho(x)}{1-F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right\}$$

with monotone $F(x)$ a solution of $F(x) + \frac{F(x)(1-F(x))F''(x)}{(F'(x))^2} = \rho(x)$ with boundary condition $F(0) = \rho_a$ and $F(1) = \rho_b$

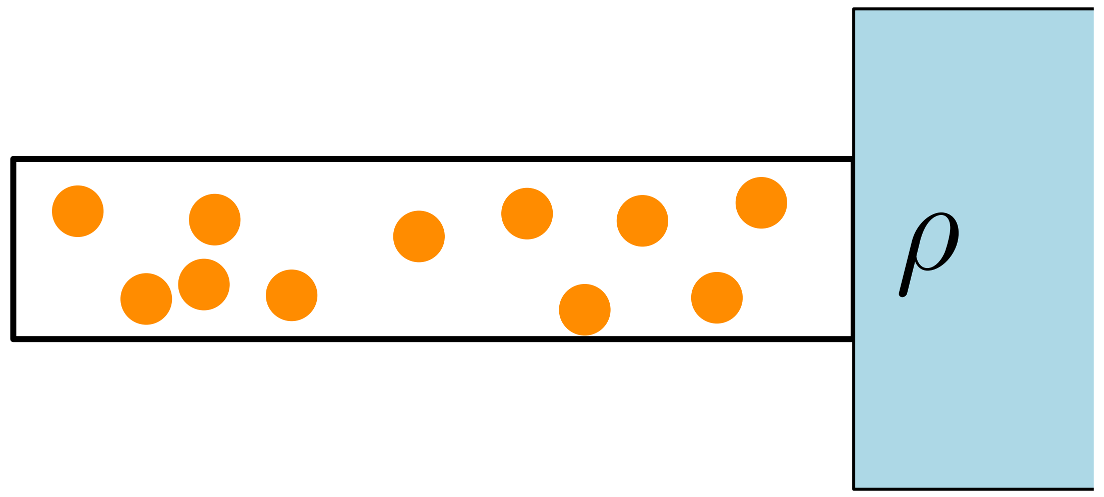
Current LDF

$$P\left(\frac{Q_t}{t} = j\right) \sim e^{-\frac{t}{L}\mathscr{R}(j\,L)} \qquad \leftrightarrow \qquad \langle e^{\lambda\, Q_t} \rangle \sim e^{\frac{t}{L}\,\mu(\lambda)}$$

$$\mu(\lambda) = \left(arcsinh \sqrt{\left(e^{\lambda} - 1\right)\rho_a(1-\rho_b) + \left(e^{-\lambda} - 1\right)\rho_b(1-\rho_a)} \right)^2$$

Why relevant? generalised free energy

Equilibrium

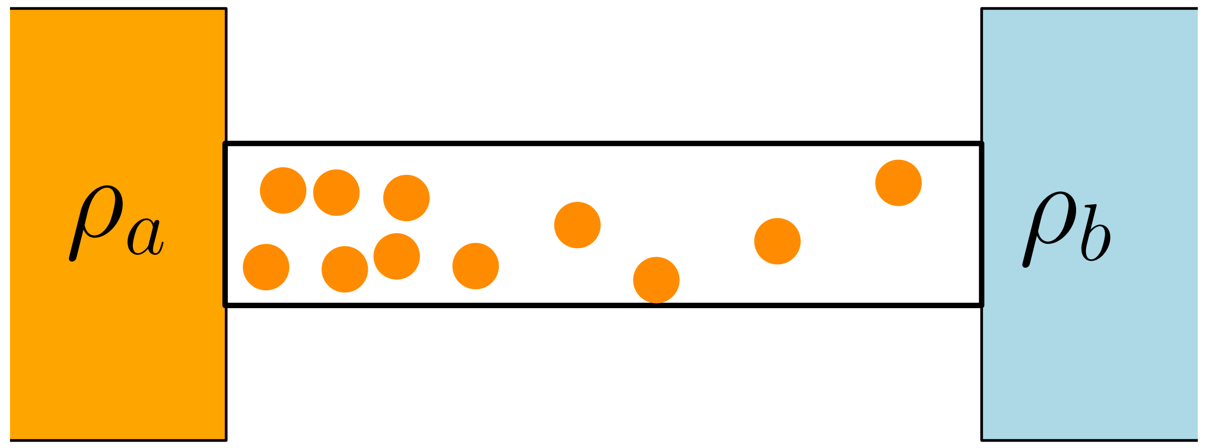


$$P_{eq}[\rho(x)] \sim e^{-L\mathscr{F}[\rho(x)]}$$

$\mathscr{F}[\rho(x)]$ **comes from statistical mechanics**

Thermodynamic state not sensitive to boundary coupling.

Non-equilibrium



$$P_{noneq}[\rho(x)] \sim e^{-L\mathscr{F}[\rho(x)]}$$

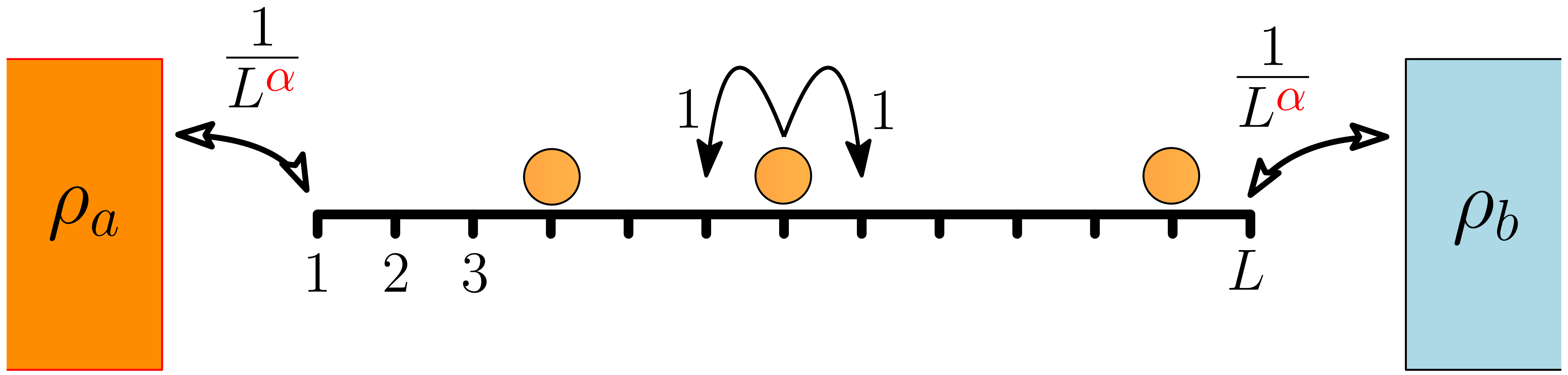
$\mathscr{F}[\rho(x)]$ **from large deviation theory**

[Kipnis Varadhan Olla 1989] [Derrida 2007] [Touchette 2009]

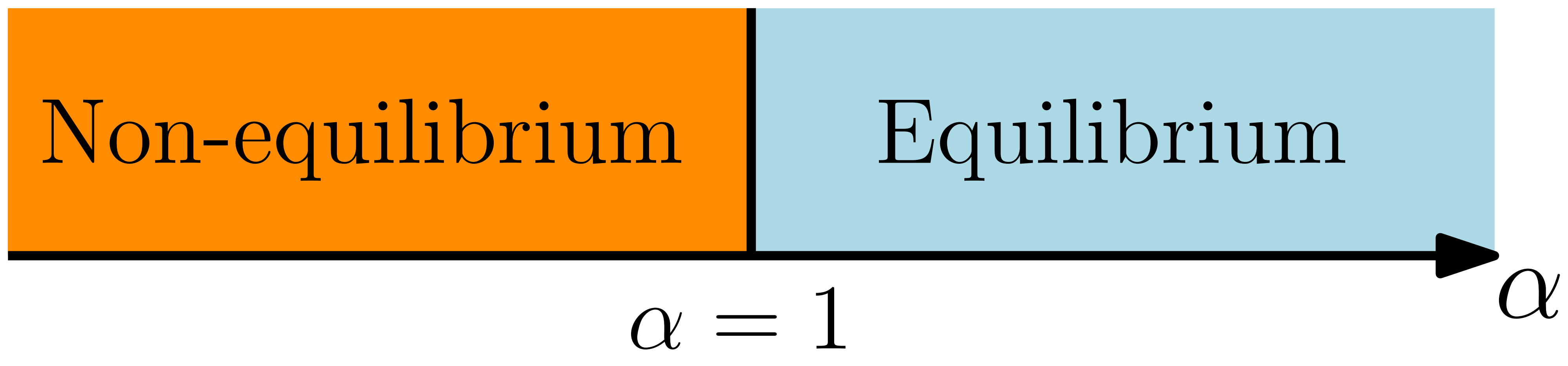
Is $\mathscr{F}[\rho(x)]$ a state-function?

(Non-trivial because of long-range correlation)

How robust is ldf to boundary perturbation?

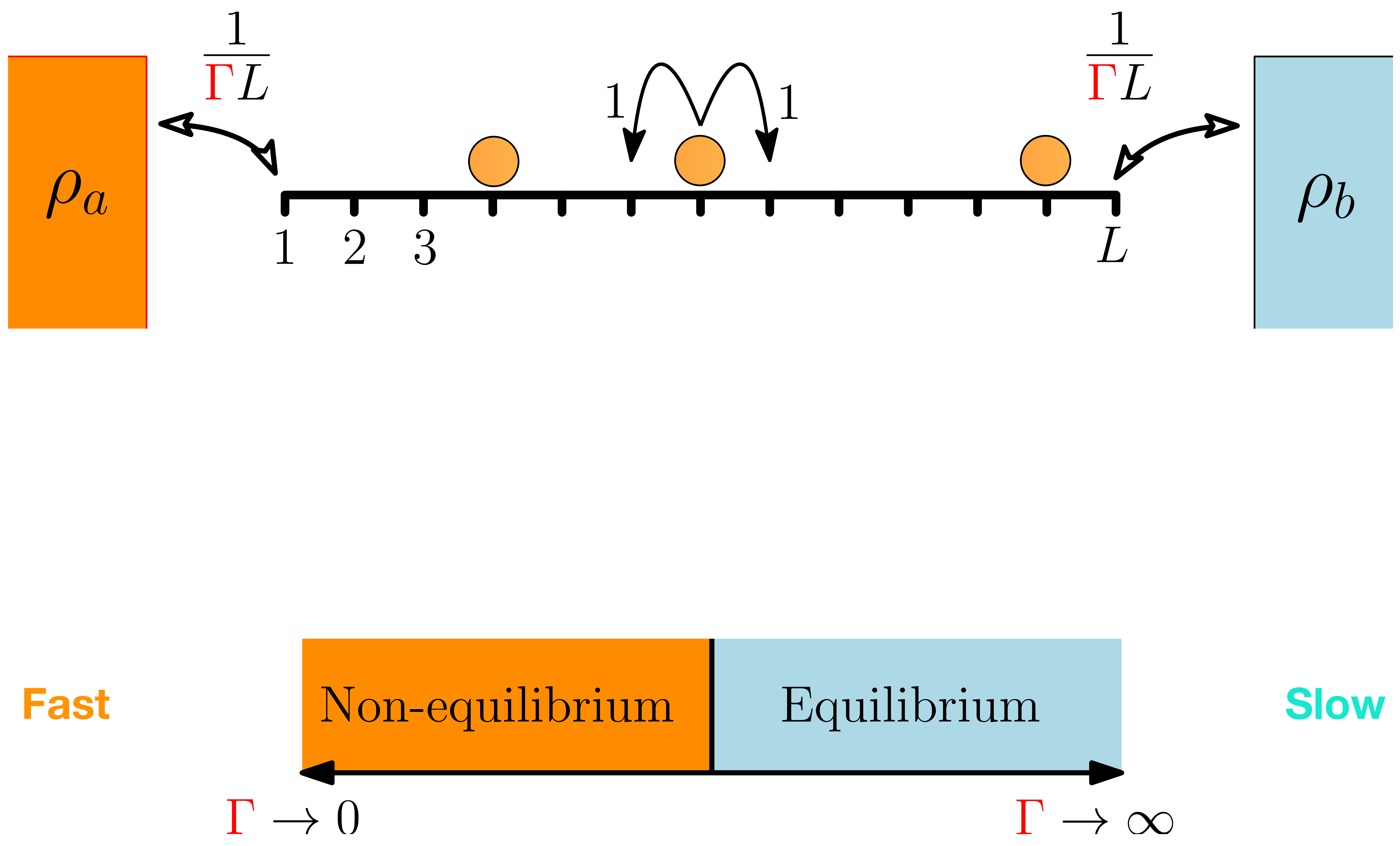


Baldasso, et al J Stat Phys 2017,
Franco et al Stoch Procc & app 2019
Franco et al Proc Math Stat 2017
Goncalvez et al Stoch Procc & app 2020
Tsunoda 2019
De Masi et al J Stat Phys 2017
Landim et al Ann Henri Poincare 2013



A single large deviation function | **A single large deviation function**

Derrida, Hirschberg, Sadhu, J stat Phys 2021


$\frac{1}{\Gamma L}$
$\frac{1}{\Gamma L}$
1
1
ρ_a
ρ_b
1
2
3
L
Fast
Non-equilibrium
Equilibrium
Slow
$\Gamma \to 0$
$\Gamma \to \infty$

Density LDF

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$$\mathscr{F}[\rho(x)] = \max_{F(x)} \int_0^1 dx \left\{ \rho(x) \ln \frac{\rho(x)}{F(x)} + (1-\rho(x)) \ln \frac{1-\rho(x)}{1-F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right\} + \Gamma \ln \frac{F(0)-\rho_a}{\Gamma(\rho_b-\rho_a)} + \Gamma \ln \frac{\rho_b - F(1)}{\Gamma(\rho_b-\rho_a)}$$

with monotone $F(x)$

Remarks

- **Non-local LDF represents long-range correlation at generic parameter values**

  [Garrido, Lebowitz, Maes, and Spohn (1990)]

- **No boundary condition on $F(x)$**

Density LDF

Derrida, Hirschberg, Sadhu, J stat Phys 2021

$$\mathcal{F}[\rho(x)] = \int_0^1 dx \left\{ \rho(x)\ln\frac{\rho(x)}{F(x)} + (1-\rho(x))\ln\frac{1-\rho(x)}{1-F(x)} + \log\frac{F'(x)}{\rho_b-\rho_a} \right\} + \Gamma\ \ln\frac{F(0)-\rho_a}{\Gamma(\rho_b-\rho_a)} + \Gamma\ \ln\frac{\rho_b - F(1)}{\Gamma(\rho_b-\rho_a)}$$

with $F(x)$ given by $$F(x) + \frac{F(x)(1-F(x))F''(x)}{(F'(x))^2} = \rho(x)$$

Non-equilibrium ($\Gamma \to 0$)	Marginal (Γ finite)	Equilibrium ($\Gamma \to \infty$)
$F(0) = \rho_a$ $F(1) = \rho_b$	$F(0) - \Gamma\, F'(0) = \rho_a$ $F(1) + \Gamma\, F'(1) = \rho_b$	$F'(0) = 0 = F'(1)$
Dirichlet condition	Robin condition	Neumann condition

Minimal of $\mathcal{F}[\rho(x)]$ is for $\bar{\rho}(x)$ which is a solution of the hydrostatic equation derived by Baldasso et al (2017) Goncalvez et al (2019, 2020).

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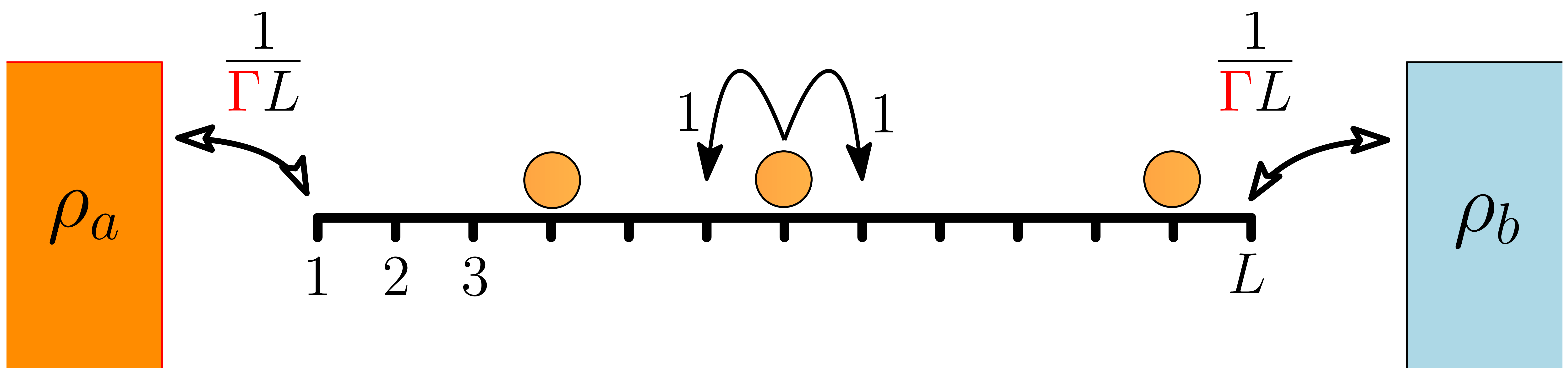
Current LDF

$$P\left(\frac{Q_t}{t} = j\right) \sim e^{-\frac{t}{L}\mathcal{R}(j\,L)} \qquad \leftrightarrow \qquad \langle e^{\lambda\,Q_t}\rangle \sim e^{\frac{t}{L}\,\mu(\lambda)}$$



$$\mu(\lambda;\rho_a,\rho_b) = \min_{t_a,t_b}\left\{ \frac{\sinh^2 t_a}{\Gamma} + \left(t_a + t_b - arcsinh\sqrt{\omega}\right)^2 + \frac{\sinh^2 t_b}{\Gamma} \right\}$$

$$\omega = \left(e^{\lambda}-1\right)\rho_a(1-\rho_b) + \left(e^{-\lambda}-1\right)\rho_b(1-\rho_a)$$

Non-equilibrium ($\Gamma \to 0$)

$$\mu(\lambda) = (arcsinh\sqrt{\omega})^2$$

[Derrida et al, 2004]

Equilibrium ($\Gamma \to \infty$)

$$\mu(\lambda) = 0$$

Consistent with exact result

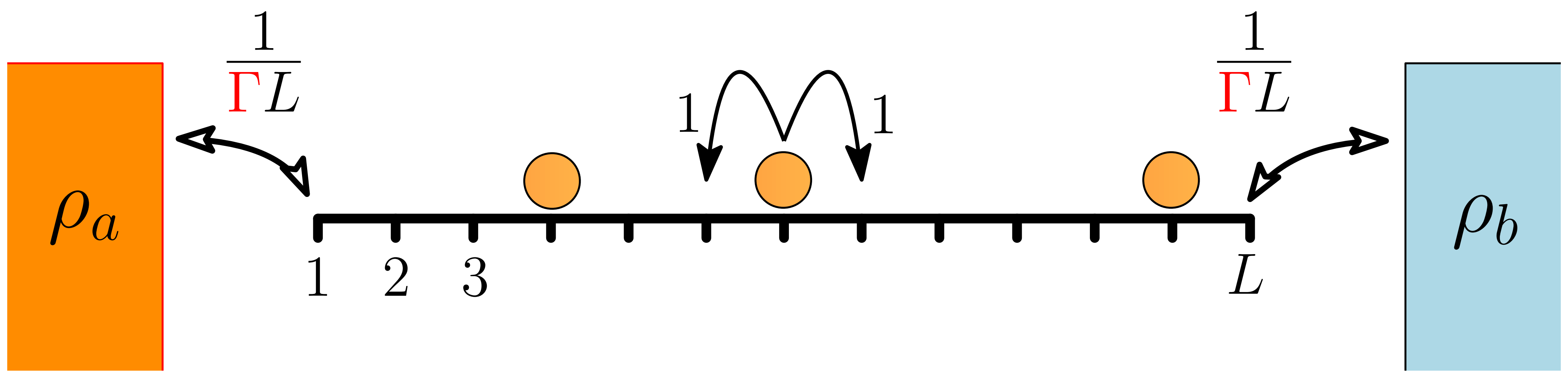
$$\mu(\lambda) = \frac{1}{L^{\alpha-1}}[-1+\sqrt{\omega+1}] \qquad for\ \alpha > 1$$

Rest of the talk

- How results are derived: matrix product formulation & tilted operator.
- Extension for larger class of systems: fluctuation hydrodynamics.

An interacting many-body problem !

Symmetric exclusion ⟷ SU(2) spin chain



$$\frac{dP_t(n_1,\ldots,n_L)}{dt} = \sum_{\mathbf{n}'} M(\mathbf{n} \leftarrow \mathbf{n}')\, P_t(\mathbf{n}')$$

Master eqn / rate eqn

$$|\psi\rangle = \sum_{\{\mathbf{n}\}} P(\mathbf{n})\,|n_1\rangle \otimes |n_2\rangle \cdots \otimes |n_L\rangle \longrightarrow \frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

$$H = \sum_{k=1}^{L-1}\left(\vec{\sigma}_k \cdot \vec{\sigma}_{k+1} - \frac{1}{4}\right) + \frac{\rho_a}{\Gamma L}\left(\sigma_1^+ + \sigma_1^z - \frac{1}{2}\right) + \frac{1-\rho_a}{\Gamma L}\left(\sigma_1^- + \sigma_1^z - \frac{1}{2}\right)$$
$$+ \frac{\rho_b}{\Gamma L}\left(\sigma_L^+ + \sigma_L^z - \frac{1}{2}\right) + \frac{1-\rho_b}{\Gamma L}\left(\sigma_L^- + \sigma_L^z - \frac{1}{2}\right)$$

- Interacting fermions. [Jordan-Wigner transformation]
- The boundary terms, make analysis difficult.

Deepak Dhar (1986)

Mustansir Berma, Stinchcombe, Grynberg (1993)]

Gwa, Spohn (1992)

Schutz and Sandow (1994)

Lazarescu, Mallick (2014)

Jan de Gier & Fabian Essler (2005)

Tailleur, Kurchan, Lecomte (2008)

How to solve this interacting many-body system?

First, the current problem

$$\langle e^{\lambda Q_T} \rangle \sim e^{\frac{T}{L}\mu(\lambda)}$$

[Derrida, Doucot, Roche 2004]

Solution by four “miracles” !

[Derrida, Doucot, Roche 2004]

[Derrida, Sadhu 2018, 2019]

Eigenvalue of tilted operator

$$\frac{dP_T}{dT} = M P_T \quad \longrightarrow \quad P_T = e^{TM} P_0$$

$$\langle e^{\lambda Q_T} \rangle = e^{T M_\lambda} P_0 \sim e^{T \mu(\lambda)}$$

Largest eigenvalue of M_λ gives $\mu(\lambda)$

A linear algebra problem!

$$[M_\lambda]_{(2^L \times 2^L)} R_{2^L} = \mu(\lambda) R_{2^L}$$

We want an exact $(\lambda, L, \rho_a, \rho_b, \Gamma)$ dependence.

Put house in order: particle sector decomposition

$$\{n_1, \cdots, n_L\} \equiv \{i_1, \cdots, i_k\} \quad \longrightarrow \quad R \equiv R(i_1, \cdots, i_k)$$

Sectors are coupled (too difficult!)

Perturbation theory in density

$$R = R_0 + R_1 + R_2 + \cdots$$

$$\mu = 0 + \mu_1 + \mu_2 + \mu_3 + \cdots$$

Breaks hierarchy.
Solve order by order.
Then, guess the full solution.

First "miracle"

R_n is an n-th degree polynomial in i

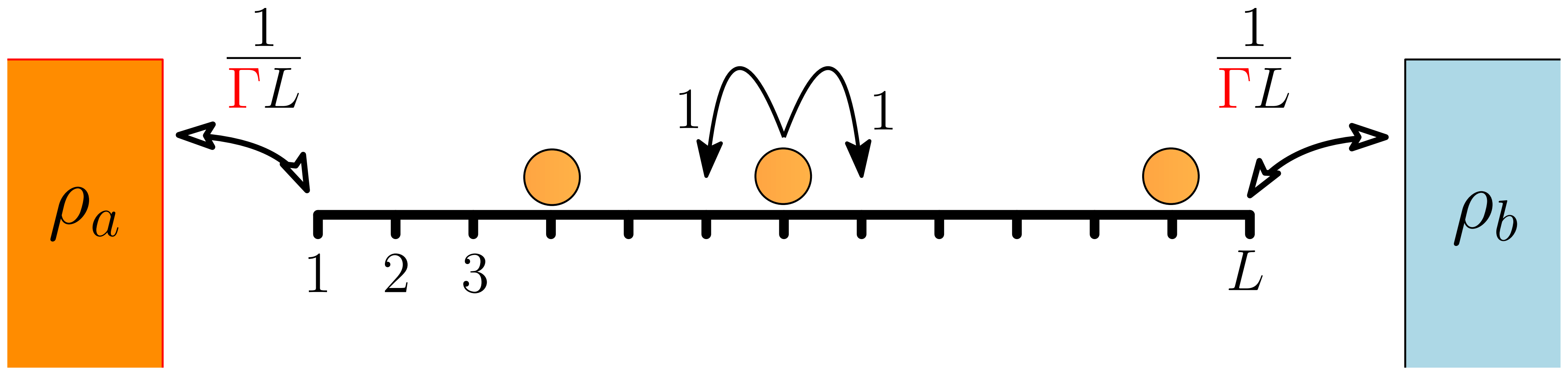
$$\mu 1 = \frac{(-1 + z)\,(z\,\rho a - \rho b)}{N\,z};$$

$N = L - 1 + \gamma_a + \gamma_b$

$z = e^{\lambda}$

[Derrida et al 2004]

Second “miracle”: symmetries



$$\mu(\lambda,\rho_a,\rho_b) = \mu(-\lambda,\rho_b,\rho_a)$$
$$= \mu(-\lambda, 1-\rho_a, 1-\rho_b)$$
$$= \mu\left(-\lambda - \ln\frac{\rho_a(1-\rho_b)}{\rho_b(1-\rho_a)}, \rho_a, \rho_b\right)$$

$$\mu(\lambda;\rho_a,\rho_b) = G(\omega) \qquad \textbf{with} \qquad \omega = \left(e^{\lambda}-1\right)\rho_a(1-\rho_b) + \left(e^{-\lambda}-1\right)\rho_b(1-\rho_a)$$

$$G|(\omega) = -\frac{1}{N}\Bigg((-\omega) + \frac{1}{3}\left(\frac{(-1+2N)}{2N} - \frac{(ab+cd)}{2(-1+N)N^2}\right)(-\omega)^2 + \frac{8}{45}\left(\frac{(-1+2N)(-1+4N)}{8N^2} - \frac{5N^2(N-3)(ab+cd) + N(ab(7+9a)+cd(7+9c)) - 5(ab+cd)^2}{8(-2+N)(-1+N)N^4}\right)(-\omega)^3 +$$
$$\frac{4}{35}\Bigg(\frac{(-1+2N)(-19+N(195-614N+432N^2))}{864(-1+N)N^3} - \frac{1}{(864(-3+N)(-2+N)(-1+N)^3N^6)}$$
$$\Big(3(-1+N)N^2(-318+2029N-4361N^2+3948N^3-1484N^4+196N^5)(ab+cd) + 36(-1+N)N^2(-1+2N)(78-98N+21N^2)(a^2b+c^2d) + -21N(90-345N+462N^2-245N^3+40N^4)(ab+cd)^2 +$$
$$60N^2(68-99N+45N^2)(a^3b+c^3d) - 175(-6+11N)(ab+cd)^3 - 126N(30-55N+24N^2)(ab+cd)(a^2b+c^2d) + 840N^2((ab+cd)^3 + 4(a^4b+c^4d) - (a^3b^3+c^3d^3))\Big)\Bigg)(-\omega)^4 + \ldots\Bigg)$$

Third "miracle": thermodynamic limit

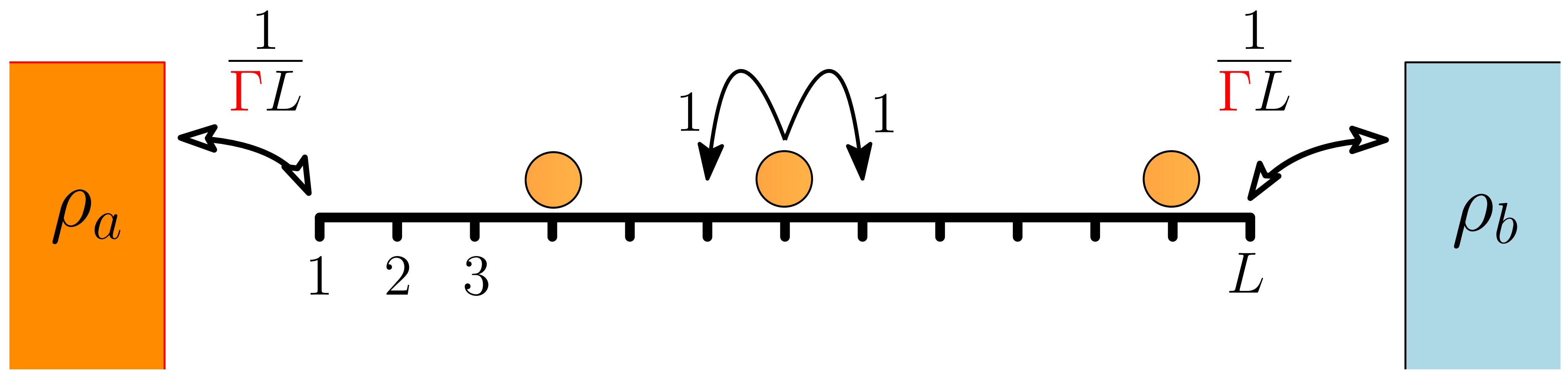
$$\mu(\lambda) = G(\omega) \simeq \frac{g(\omega)}{L\,(1+\Gamma+\Gamma)}$$

$$g = (-\omega) + \frac{1}{3}\left(1-\left(\ell 1^3+\ell 2^3\right)\right)(-\omega)^2 + \frac{8}{45}\left(1-\frac{5\left(\ell 1^3+\ell 2^3\right)-10\left(\ell 1^3+\ell 2^3\right)^2+9\left(\ell 1^5+\ell 2^5\right)}{4}\right)(-\omega)^3 + \frac{4}{35}\left(1-\frac{1}{36}\left(49\left(\ell 1^3+\ell 2^3\right)-140\left(\ell 1^3+\ell 2^3\right)^2+280\left(\ell 1^3+\ell 2^3\right)^3+126\left(\ell 1^5+\ell 2^5\right)+225\left(\ell 1^7+\ell 2^7\right)-504\left(\ell 1^3+\ell 2^3\right)\left(\ell 1^5+\ell 2^5\right)\right)\right)(-\omega)^4$$

Fourth "miracle": time-scale

For $\Gamma \to 0$
Fast coupling

$$g = I(-\omega) + \frac{1}{3}(-\omega)^2 + \frac{8}{45}(-\omega)^3 + \frac{4}{35}(-\omega)^4 + \ldots$$



For non-zero Γ the system may be considered in three "independent" parts.

$$P(j) \simeq \max_{\rho_0,\rho_1} P_{\text{left}}(j,\rho_a,\rho_0)\, P_{\text{bulk}}(j,\rho_0,\rho_1)\, P_{\text{right}}(j,\rho_1,\rho_b)$$

Additivity conjecture

Bodineau & Derrida 2004

$$\mu(\lambda;\rho_a,\rho_b) = \max_{\rho_0,\rho_1}\min_{\lambda_0,\lambda_1}\left\{\frac{\mu_{bond}(\lambda_0,\rho_a,\rho_0)}{\Gamma_a} + \mu_{bulk}(\lambda-\lambda_0-\lambda_1,\rho_0,\rho_1) + \frac{\mu_{bond}(\lambda_1,\rho_1,\rho_b)}{\Gamma_b}\right\}$$

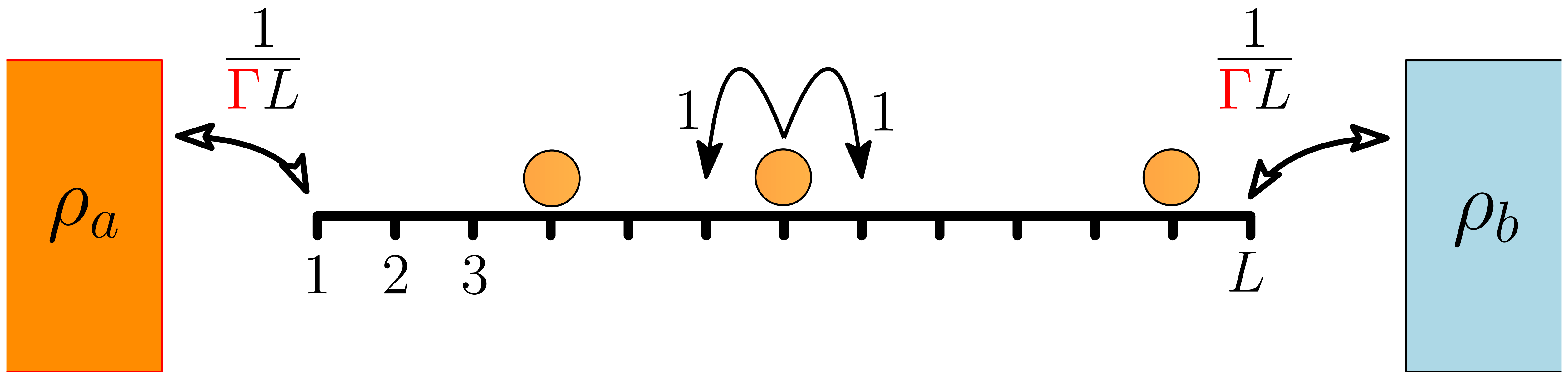
$$g(\omega(\lambda;\rho_a,\rho_b)) = \max_{\rho_0,\rho_1}\min_{\lambda_0,\lambda_1}\left\{\frac{\omega(\lambda_0,\rho_a,\rho_0)}{\Gamma_a} + \left(arcsinh\sqrt{\omega(\lambda-\lambda_0-\lambda_1,\rho_0,\rho_1)}\right)^2 + \frac{\omega(\lambda_1,\rho_1,\rho_b)}{\Gamma_b}\right\}$$

$$\omega(\lambda,x,y) = \left(e^{\lambda}-1\right)x(1-y) + \left(e^{-\lambda}-1\right)y(1-x)$$

$$g(\omega) = -\min_{t_a,t_b}\left\{\frac{\sinh^2 t_a}{\Gamma_a} + \left(t_a+t_b-arcsinh\sqrt{\omega}\right)^2 + \frac{\sinh^2 t_b}{\Gamma_b}\right\}$$

Now, the density LDF

$$P[\rho(x)] \sim e^{-L\,\mathscr{F}[\rho(x)]}$$



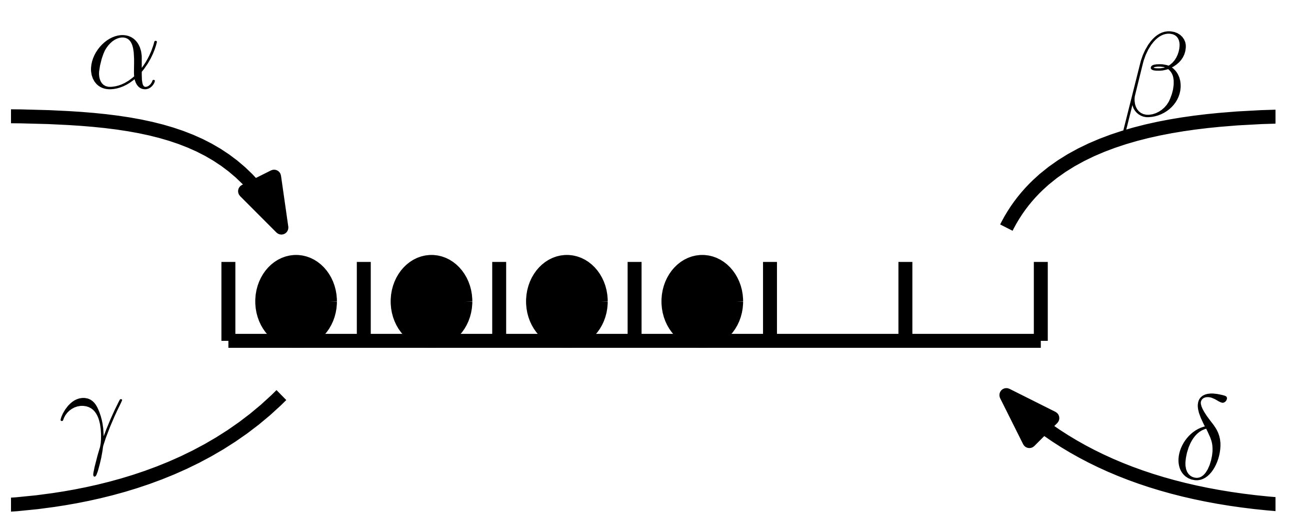
No Equilibrium Stat-mech to help.

$$P(C) \neq e^{-\beta\, E(c)}$$

What is the $P(C)$?

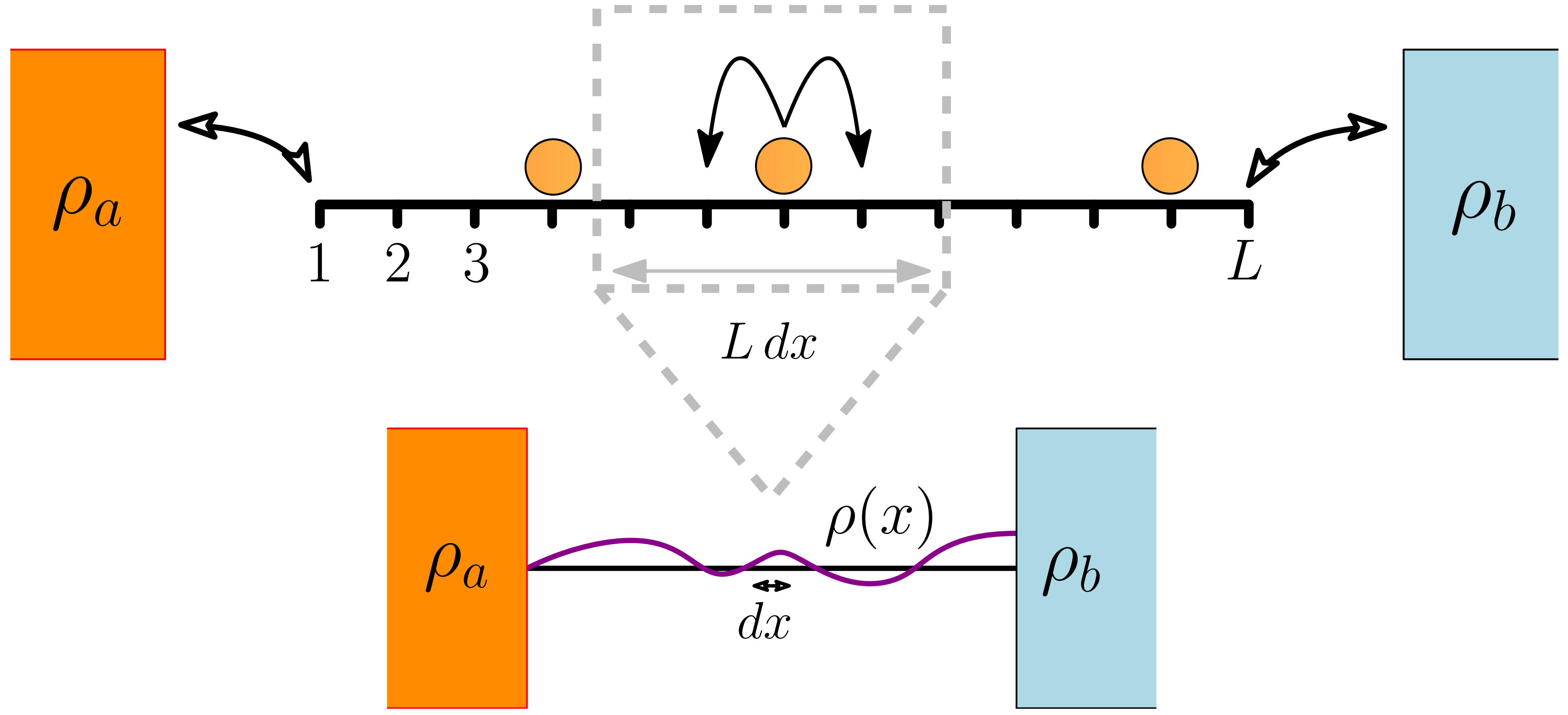
Matrix product representation

[Derrida, Evans, Hakim, Pasquier, 1993] [Blythe, Evans, 2007]



$$P(C) \propto \langle L | WWWWWZZ | R \rangle$$

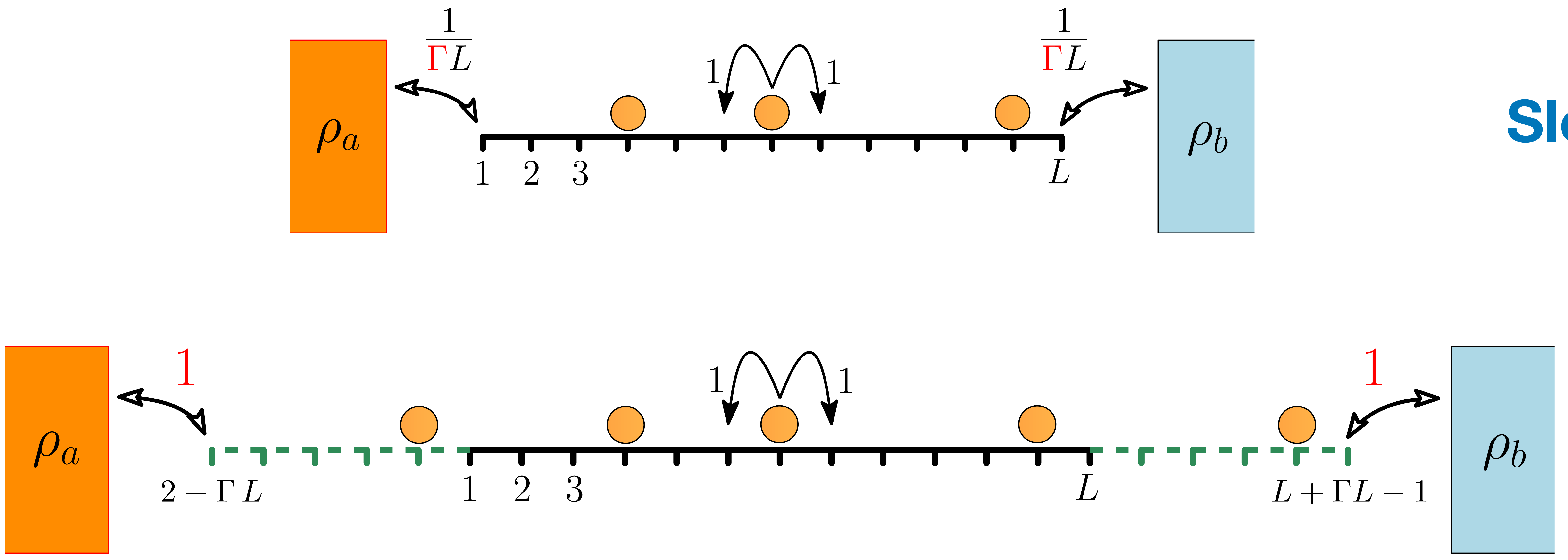
W and Z follows quadratic algebra



$$P[\rho(x)] \sim e^{-L\,\mathcal{F}[\rho(x)]}$$

Derrida, Lebowitz, Speer (PRL, 2000) (JSP 2002)

Physical picture



Slow coupling.

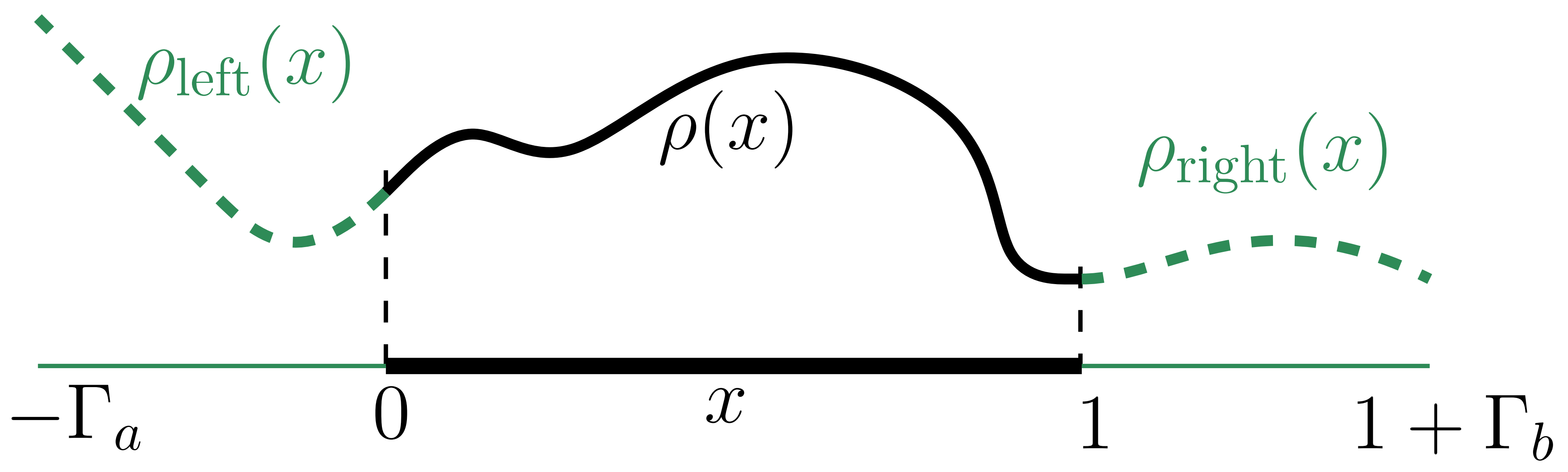
Fast coupling.

$$\langle n_{i_1} \cdots n_{i_k} \rangle_L^{(weak)} = \langle n_{i_1} \cdots n_{i_k} \rangle_{L+\Gamma_a L+\Gamma_b L}^{(strong)}$$

for all correlations !

[Derrida, Lebowitz, Speer (2007)]

Macroscopic scale



$$P[\rho(x)] = \int \mathscr{D}[\rho_{left}, \rho_{right}]\, P_{strong}[\rho_{left}, \rho, \rho_{right}] \sim \mathbf{e}^{-\mathbf{L}\,\mathscr{F}[\rho(\mathbf{x})]}$$

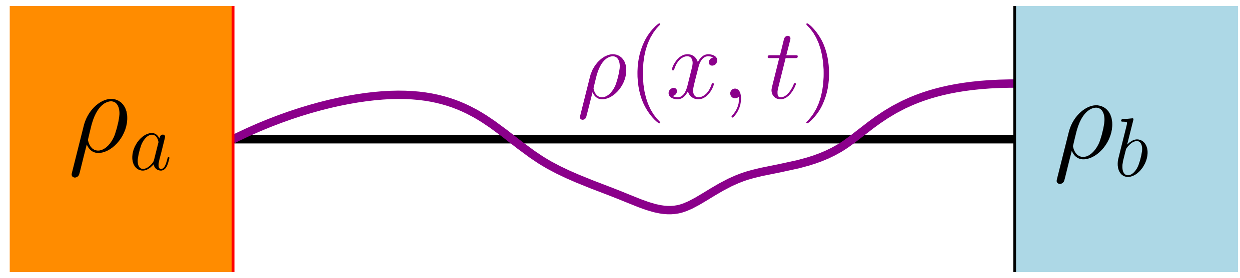
$$\mathscr{F}[\rho(x)] = \max_{F(x)} \int_0^1 dx \left\{ \rho(x) \ln \frac{\rho(x)}{F(x)} + (1-\rho(x)) \ln \frac{1-\rho(x)}{1-F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right\} + \Gamma \ln \frac{F(0)-\rho_a}{\Gamma(\rho_b-\rho_a)} + \Gamma \ln \frac{\rho_b - F(1)}{\Gamma(\rho_b - \rho_a)}$$

What’s the point of exact results?

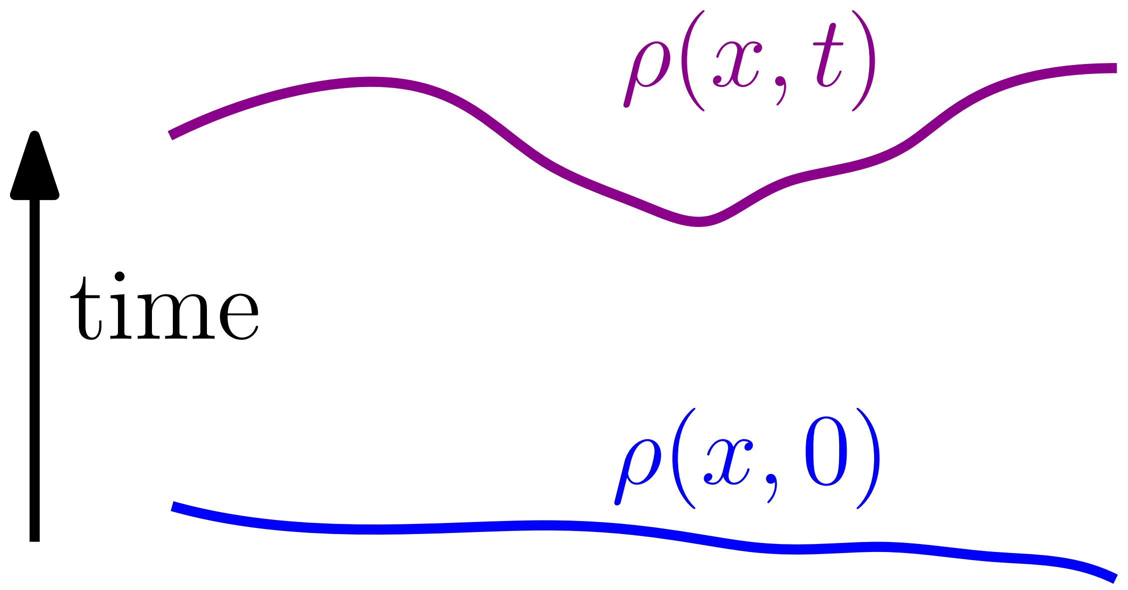
To test a “less exact”, but general, powerful approach.

A fluctuating hydrodynamic approach

Fluctuating hydrodynamics



Coarse grained



Path-space measure

$$P[\rho(t), j(t)] \sim e^{-S[\rho, j]}$$

Action

Macroscopic fluctuation theory,

Bertini, de Sole, Gabrielli, Jona-Lassinio, Landim (2001) … (2016) Rev Mod Phys.

How to get the Action?

Going beyond small fluctuations

Getting the hydrodynamic Action

Mathematical approach.

Bertini, De-Sole, Gabrielli, Jona-Lasinio, Landim 2001 —- 2015
Book by Kipnis & Landim 1999
Patricia Goncalves, Stefano Olla, Cedric Bernardin, De Masi …

Via spin chain

Coherent state path integral method

Tailleur, Kurchan, Lecomte 2008

Phenomenological approach.

$$\partial_t \rho(x,t) = -\partial_x j(x,t); \qquad j(x,t) = E[\rho(x,t)] + \frac{1}{\sqrt{L}}\sqrt{\sigma(\rho)}\ \xi(x,t)$$

Martin-Siggia-Rose-Janssen-de-Dominicis formalism

Derrida 2007
Tailleure, Kurchan, Lecomte 2007
Krapivsky, Meerson 2012
Krapivsky, Mallick, Sadhu 2014

A rudimentary way …

Stochastic dynamics → **Microscopic Action**

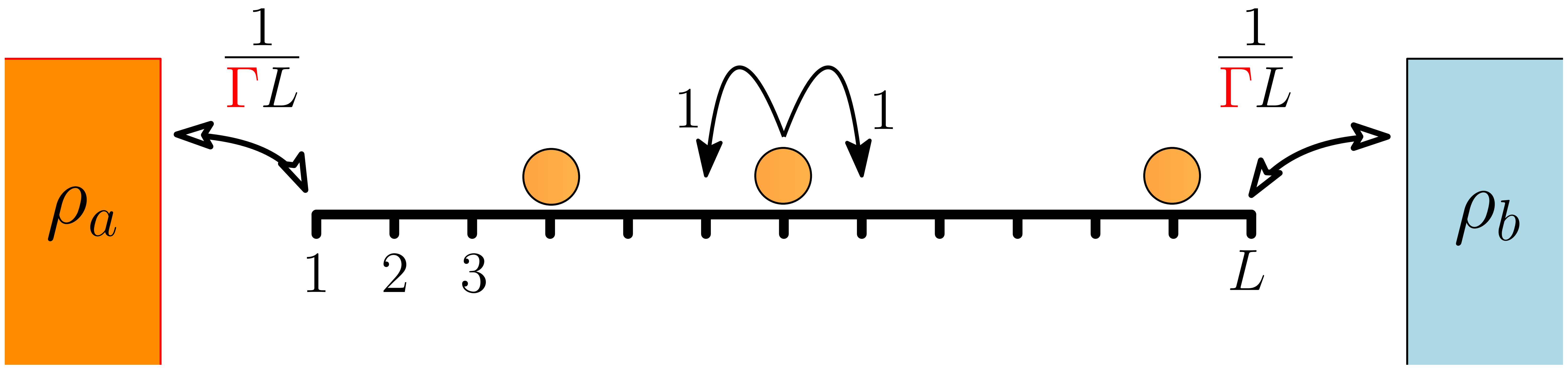
↓

Hydrodynamic Action

Saha & TS 2023
Lefevre & Biroli 2007.

What is the Action for SEP?

Action for SEP



$$S[\rho, \hat{\rho}] = \int dt \left\{ \int_0^1 dx\, \hat{\rho}\, \dot{\rho} - H_{left} - H_{bulk} - H_{right} \right\}$$

Conjugate of $j(x,t)$

$$H_{bulk} = \int_0^1 dx \left\{ \rho(1-\rho)(\partial_x \hat{\rho})^2 - \partial_x \rho\, \partial_x \hat{\rho} \right\}$$

$$H_{left} = \frac{1}{\Gamma} \left\{ \left(e^{\hat{\rho}_0} - 1\right) \rho_a (1 - \rho_0) + \left(e^{-\hat{\rho}_0} - 1\right) \rho_0 (1 - \rho_a) \right\}$$

A side remark

$$\Pr\left[\rho(x,t), j(x,t)\right] \propto \left[\prod_{t,x} \delta\big(\partial_t \rho + \partial_x j\big)\right] \mathrm{e}^{-L\int_{-\infty}^{0} \mathrm{d}t\, \mathscr{L}[j,\rho]}$$

$$\mathscr{L} = \int_0^1 \mathrm{d}x\, \frac{\big(j(x,t) + \partial_x \rho(x,t)\big)^2}{4\,\rho(x,t)\,(1-\rho(x,t)\big)} + \Phi_{\mathrm{lft}}\big(j(0,t)\,,\rho(0,t)\big) + \Phi_{\mathrm{rgt}}\big(j(1,t)\,,\rho(1,t)\big)$$

$$\Phi_{\mathrm{lft}}(j,\rho) = \min_{\hat\rho}\left(\hat\rho\, j - \frac{\omega(\hat\rho\,,\rho_a\,,\rho)}{\Gamma_a}\right), \quad \Phi_{\mathrm{rgt}}(j,\rho) = \min_{\hat\rho}\left(\hat\rho\, j - \frac{\omega(\hat\rho\,,\rho\,,\rho_b)}{\Gamma_b}\right)$$

Can we reproduce the density LDF result?

Yes!

[Bouley, Erignoux, Landim 2021]

Soumyabrata Saha and TS 2023

$$P[r(x)] = \int_{\bar{\rho}}^{r(x)} \mathcal{D}[\rho, \hat{\rho}] e^{-L\, S[\rho, \hat{\rho}]} \sim e^{-L\, S_{min}[r(x)]} = e^{-L\, \mathcal{F}[r(x)]}$$

Saddle point / Least Action calculation

For density

$$\mathscr{F}[r(x)] = S_{min}[r(x)]$$

Euler-Lagrange Equation

$$\partial_t \rho = \partial_x^2 \rho - \partial_x \left[2\rho(1-\rho)\partial_x \hat{\rho} \right]$$
$$\partial_t \hat{\rho} = -\partial_x^2 \hat{\rho} - (1-2\rho)\left(\partial_x \hat{\rho}\right)^2$$

Temporal boundary condition

$$\rho(x,0) = r(x)$$
$$\rho(x,-\infty) = \bar{\rho}(x)$$

Spatial boundary condition

$$\left.\begin{aligned} \Gamma\, \partial_x \hat{\rho} &= \left(e^{\hat{\rho}} - 1\right)\rho_a - \left(e^{-\hat{\rho}} - 1\right)(1-\rho_a) \\ \Gamma \left[2\rho(1-\rho)\partial_x \hat{\rho} - \partial_x \rho\right] &= e^{\hat{\rho}}\rho_a(1-\rho) - e^{-\hat{\rho}}\rho(1-\rho_a) \end{aligned}\right\} \text{for } x=0$$

Strong coupling solved by Non-local transformation [Tailleur, Kurchan, Lecomte (2007) Phys Rev Lett]

Relation to classical integrability [Mallick, Moriya, Sasamoto (2022), Bettelheim, Smith, Emerson (2021)]

A solution

A simple local change of variable

$$\hat{\rho} = \log \frac{\rho(1-F)}{F(1-\rho)}$$

[Derrida (2011)]

Euler-Lagrange Equation

$$\partial_t \rho(x,t) + \partial_x^2 \rho(x,t) = \partial_x \left[2\rho(x,t)\left(1-\rho(x,t)\right) \frac{\partial_x F(x,t)}{F(x,t)\left(1-F(x,t)\right)} \right]$$

$$\partial_t F(x,t) + \partial_x^2 F(x,t) = 2\left[\partial_x^2 F(x,t) - \frac{\rho(x,t)-F(x,t)}{F(x,t)\left(1-F(x,t)\right)} \left(\partial_x F(x,t)\right)^2 \right]$$

Temporal boundary condition

$$\rho(x,0) = r(x)$$

$$\rho(x,-\infty) = \bar{\rho}(x)$$

Spatial boundary condition

$$F(0,t) - \Gamma\, \partial_x F(0,t) = \rho_a$$

$$\rho(0,t) - \Gamma\, \partial_x \rho(0,t) = \rho_a + \frac{\left(1-2\rho(0,t)\right)\left(F(0,t)-\rho(0,t)\right)\left(F(0,t)-\rho_a\right)}{F(0,t)\left(1-F(0,t)\right)}$$

} **for x=0**

A solution

The solution of the Euler-Lagrange equation

$$\partial_t F + \partial_x^2 F = 0; \qquad F(0,t) - \Gamma \partial_x F(0,t) = \rho_a; \qquad F(1,t) + \Gamma \partial_x F(1,t) = \rho_b$$

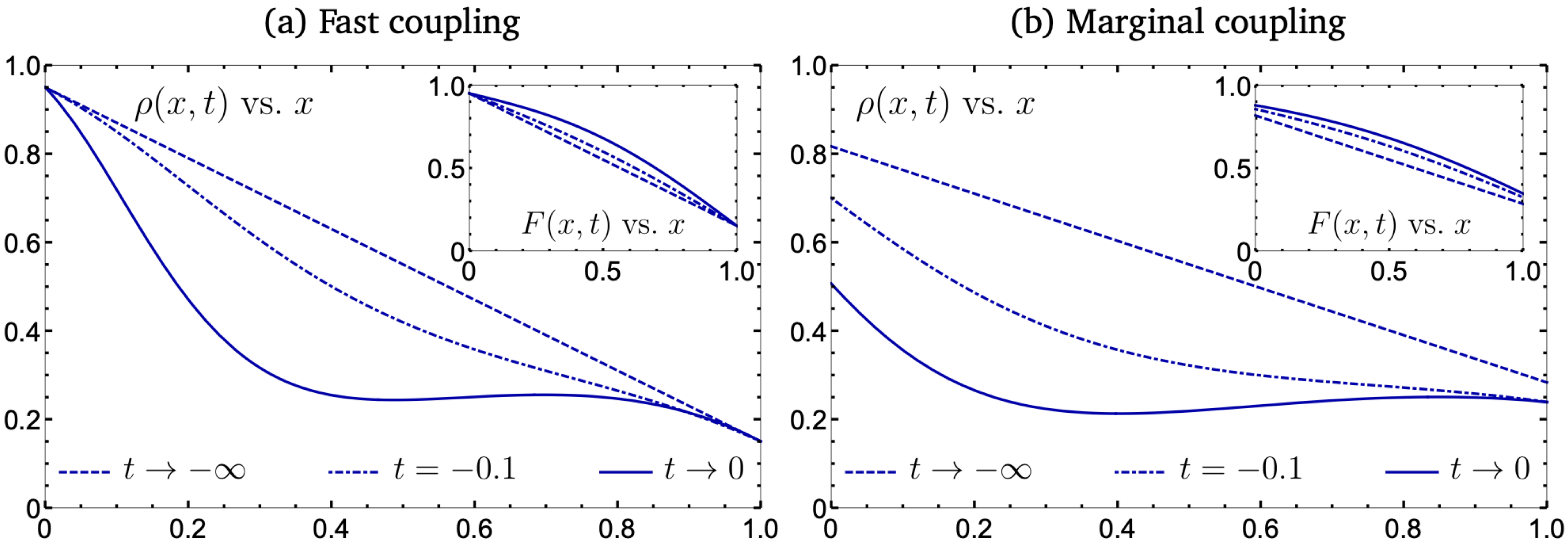
$$\rho = F + \frac{F(1-F)\partial_x^2 F}{(\partial_x F)^2}$$

This is a particular solution for the initial condition $\rho(x, -\infty) = \bar{\rho}(x)$

The minimal Action corresponding to the solution gives the LDF

Soumyabrata Saha and TS 2023

Also gives the path to a fluctuation



A side note

Hamilton-Jacobi equation

$$H\left[\frac{\delta\mathcal{F}[r(x)]}{\delta r(x)}, r(x)\right] = 0$$

Can we reproduce the result for current?

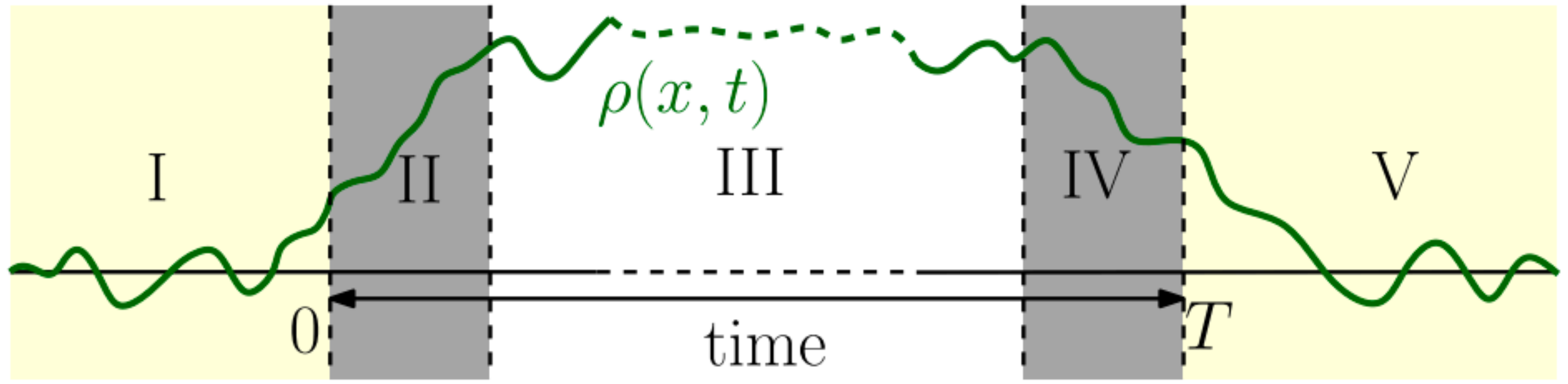
Yes!

Soumyabrata Saha and TS 2023

$$\langle e^{\lambda Q_T} \rangle = \int \mathscr{D}[\rho, \hat{\rho}] e^{-L\, S_\lambda[\rho,\hat{\rho}]} \sim e^{-L\, \min S_\lambda}$$

Saddle point / Least Action calculation

The minimal-Action path is time-independent



$$\left[2\rho(x)\left(1-\rho(x)\right)h'(x)-\rho'(x)\right]'=0$$

$$\left(1-2\rho(x)\right)\left(h'(x)\right)^2+h''(x)=0$$

With boundary condition at x=0

$$\Gamma\, h'(0)=\rho_a\left(\mathrm{e}^{h(0)}-1\right)-(1-\rho_a)\left(\mathrm{e}^{-h(0)}-1\right)$$

$$\Gamma\left[2\rho(0)\left(1-\rho(0)\right)h'(0)-\rho'(0)\right]=\rho_a\left(1-\rho(0)\right)\mathrm{e}^{h(0)}-\rho(0)\,(1-\rho_a)\,\mathrm{e}^{-h(0)}$$

A parametric solution

$$\rho(x) = \frac{1}{2}\left(1 + \frac{\sinh\left\{2\left[\theta_a + (\theta_b - \theta_a)\,x\right]\right\}}{\sinh(2f)}\right)$$

$$h(x) = \mathrm{c} + \log\left[\frac{\cosh\left(f - \theta_a - (\theta_b - \theta_a)\,x\right)}{\cosh\left(f + \theta_a + (\theta_b - \theta_a)\,x\right)}\right]$$

with the parameters determined from the four boundary conditions

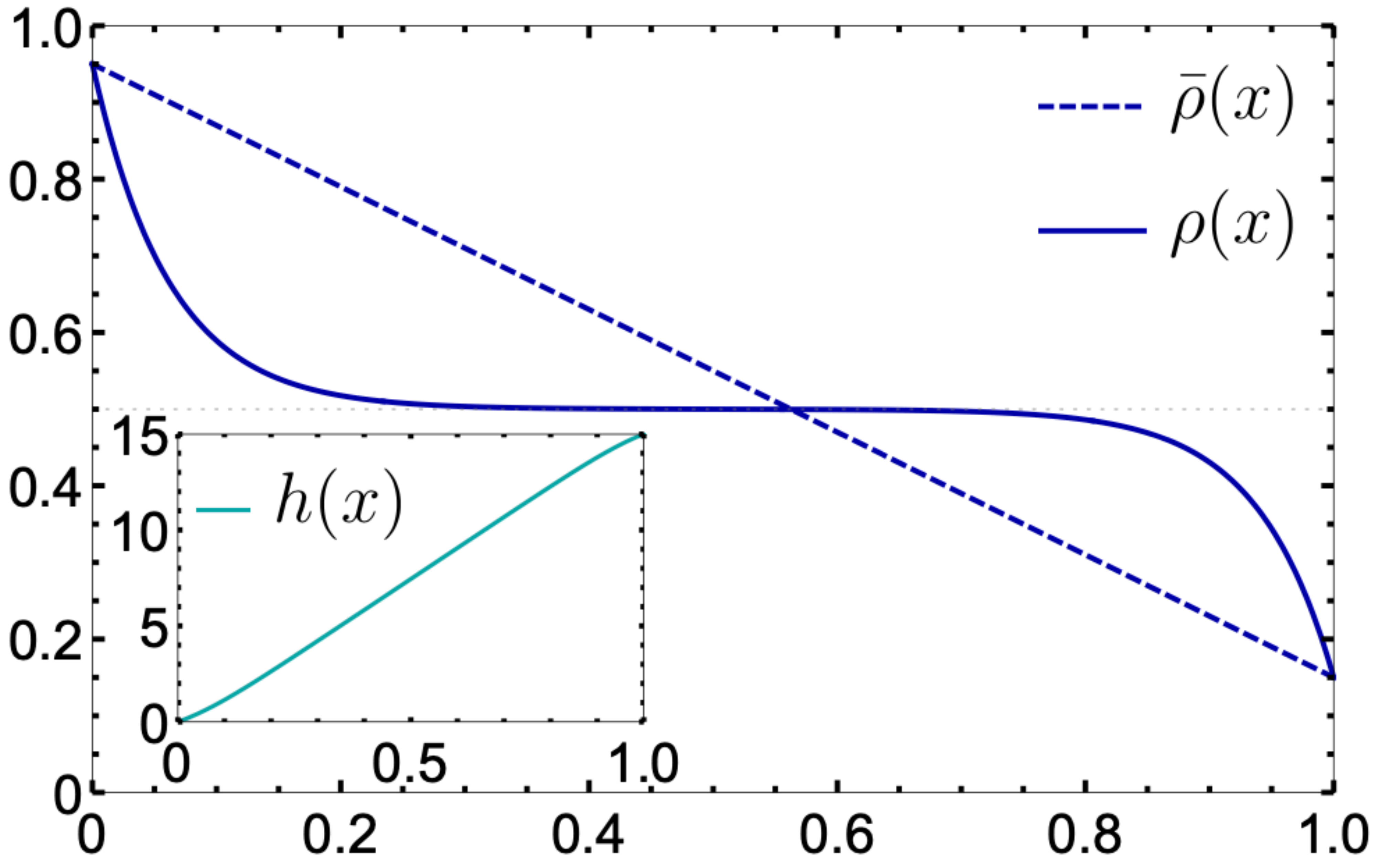
[Bodineau and Derrida 2004]

Minimal action gives a parametric form of the SCGF of current

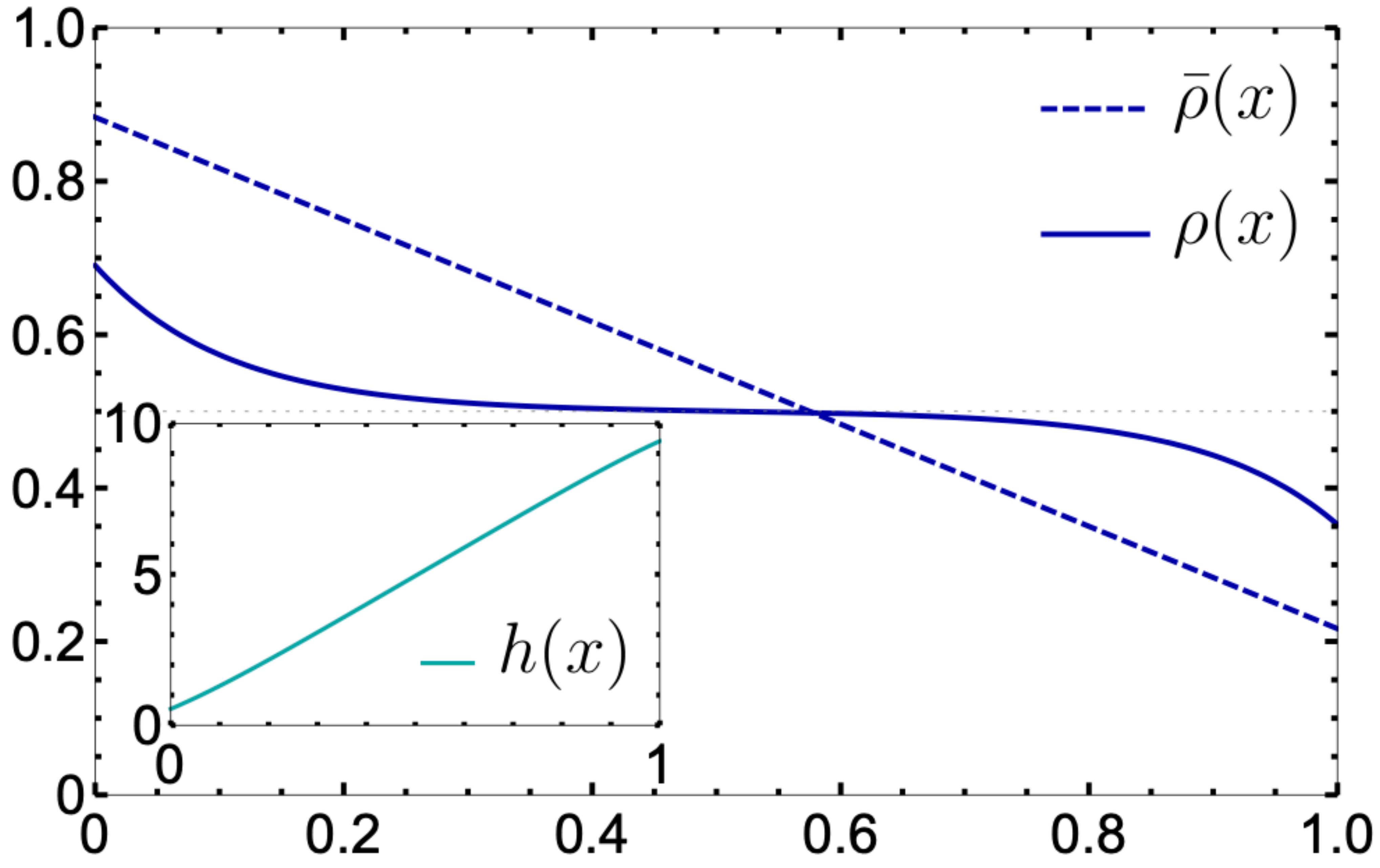
$$g(\omega) = \theta^2 + \frac{\sqrt{1 + 4\,\theta^2\,\Gamma^2} - 1}{\Gamma}$$

$$\omega = \sinh^2\theta + 2\Gamma\,\theta\,\sinh(2\theta)\sqrt{1 + 4\,\theta^2\,\Gamma^2} + \left(-1 + 4\,\Gamma^2\theta^2 + \frac{1 + 4\,\theta^2\,\Gamma^2}{2}\right)\cosh(2\theta)$$

Gives the optimal quasi-stationary density profile



Fast coupling



Marginal coupling

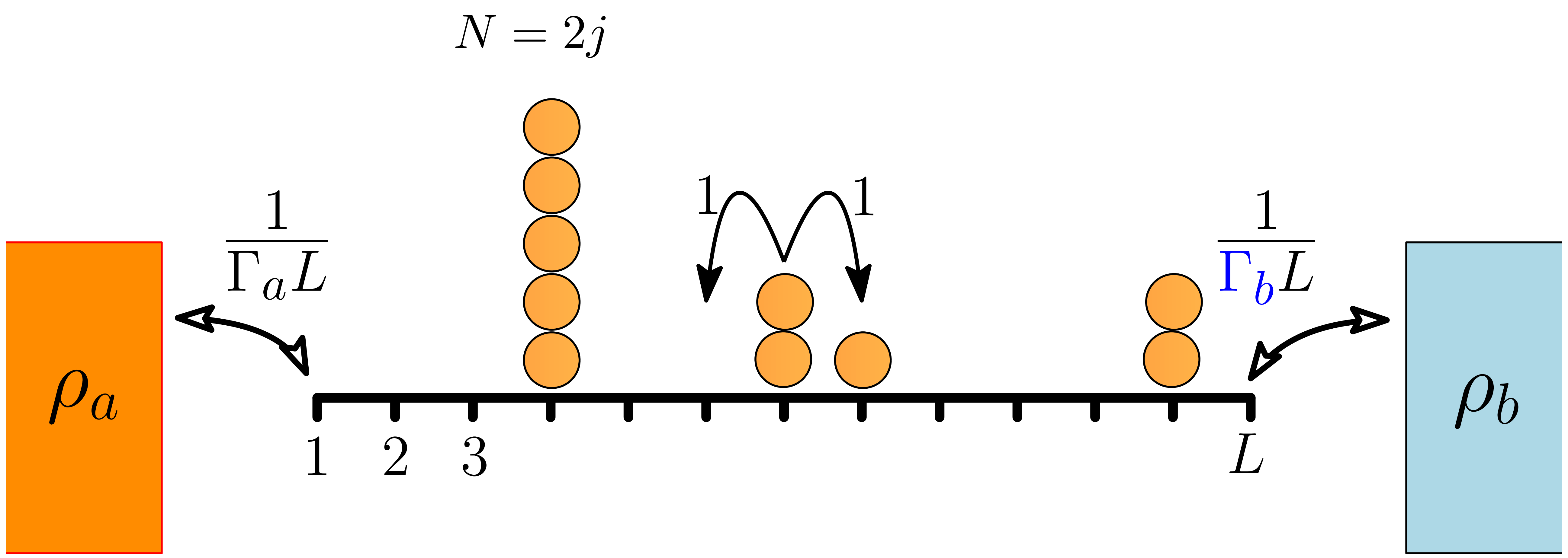
What is the hydrodynamics good for?

Generality.

Generalised/partial SEP

Tailleur, Kurchan, Lecomte 2007

Frassek, Giardina, Kurchan 2020

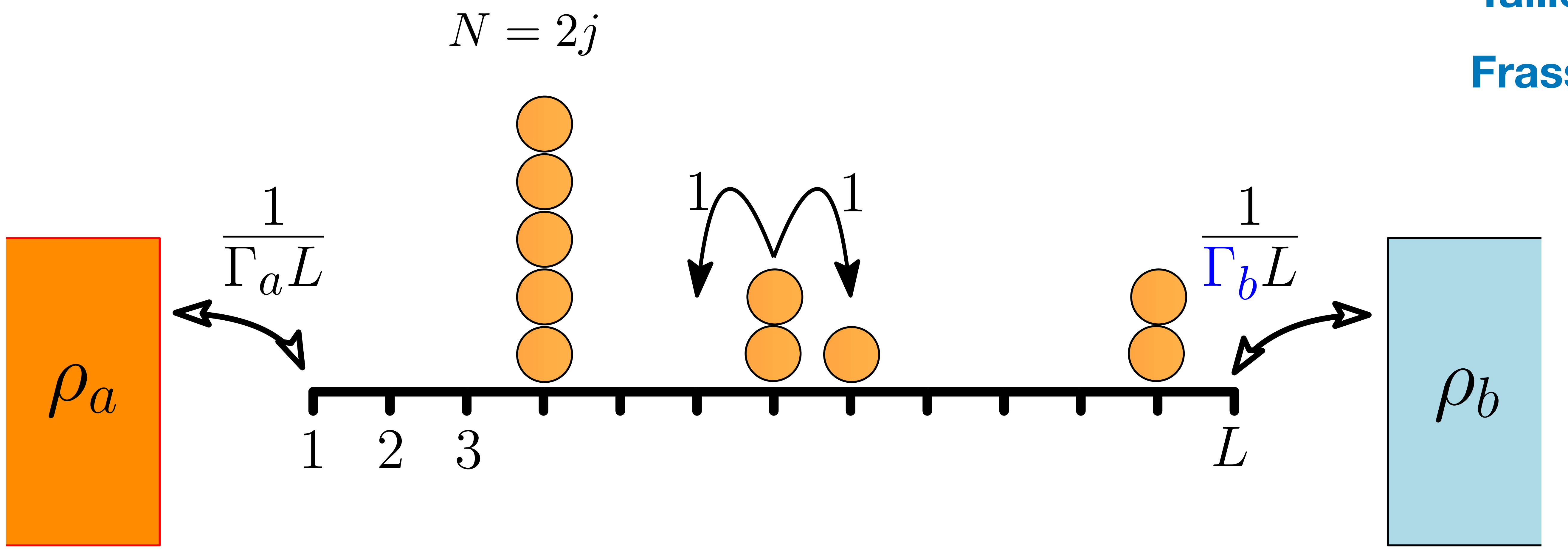


A non-integrable spin chain!

Non-integrable SU(2) spin chain

Tailleur, Kurchan, Lecomte 2007

Frassek, Giardina, Kurchan 2020



$$H = \frac{1}{2j}\sum_k \left(\vec{\sigma}_k \cdot \vec{\sigma}_{k+1} - j^2\right) + \frac{\rho_a}{\Gamma_a L}\left(\sigma_1^+ + \sigma_1^z - j\right) + \frac{1-\rho_a}{\Gamma_a L}\left(\sigma_1^- + \sigma_1^z - j\right)$$

$$+ \frac{\rho_b}{\Gamma_b L}\left(\sigma_L^+ + \sigma_L^z - j\right) + \frac{1-\rho_b}{\Gamma_b L}\left(\sigma_L^- + \sigma_L^z - j\right)$$

$$[\sigma^z, \sigma^\pm] = \pm\,\sigma^\pm \qquad [\sigma^+, \sigma^-] = 2\sigma^z$$

$$\sigma^z = \begin{pmatrix} -j & 0 & \cdots & 0 \\ 0 & -j+1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & j \end{pmatrix} \quad \sigma^+ = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2j & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

An N-dimensional representation of SU(2) group

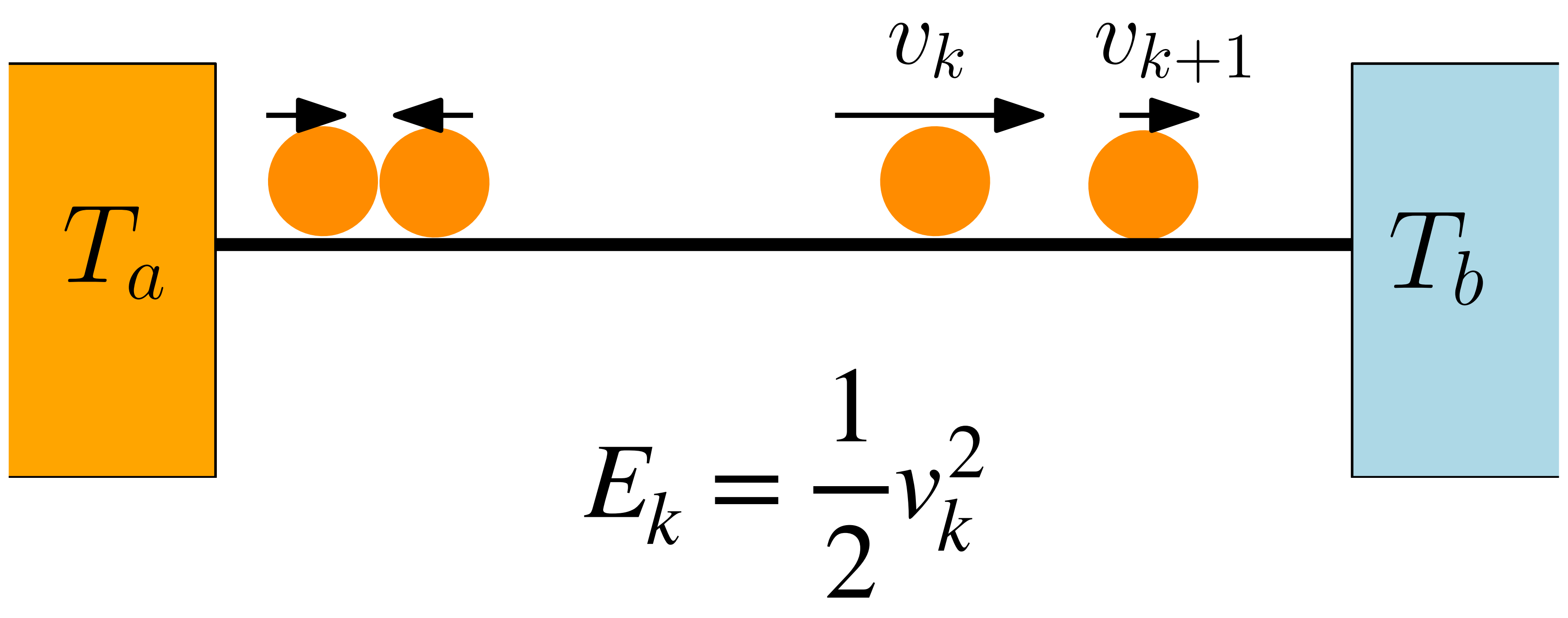
A non-unitary representation as σ^+ is not adjoint of σ^-

Similar hydrodynamics as in SEP

Integrable hydrodynamics !

Kipnis Marchioro Pressutti model

Bernardin, and Olla 2005
Frassek, Giardina, Kurchan 2020



$$\{E_k, E_{k+1}\} \longrightarrow \{p(E_k + E_{k+1}), (1-p)(E_k + E_{k+1})\}$$

Only Energy preserving collisions

A non-integrable microscopic dynamics.

How to see SU(1,1)?

Bernardin, and Olla 2005

Frassek, Giardina, Kurchan 2020

Define $S^+ = v^2$ $S^- = \frac{\partial^2}{\partial v^2}$ $S^z = \left(v\frac{\partial}{\partial v} + \frac{\partial}{\partial v} v \right)$

$$M = \sum_{k=1}^{L-1} \left(S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ - 2 S_k^z S_{k+1}^z + \frac{1}{4} \right) + \frac{1}{\Gamma_a L} \left(T_a S_1^- - S_1^z + \frac{1}{4} \right) + \frac{1}{\Gamma_b L} \left(T_a S_L^- - S_L^z + \frac{1}{4} \right)$$

Lie algebra

$$\left[S^z, S^\pm \right] = \pm S^\pm \qquad \left[S^+, S^- \right] = -2S^z$$

Non-integrable SU(1,1)

Hydrodynamics is strangely similar!

$$S[\rho, \hat{\rho}] = \int dt \left\{ \int_0^1 dx\, \hat{\rho}(x,t) \frac{\partial \rho(x,t)}{\partial t} - H_{left} - H_{bulk} - H_{right} \right\}$$

$$H_{bulk} = \int_0^1 dx \left\{ \frac{\sigma(\rho)}{2} (\partial_x \hat{\rho})^2 - \partial_x \rho\ \partial_x \hat{\rho} \right\} \qquad \text{with} \qquad \sigma(\rho) = 2\rho^2$$

$$H_{left} = \frac{1}{\Gamma_a} \left(\frac{e^{-\hat{\rho}_0 \rho_0}}{1 - T_a \hat{\rho}_0} - 1 \right)$$

Integrable hydrodynamics !

How about even more general systems?

Diffusive systems with single conserved field

$$S[\rho,\hat{\rho}] = \int dt \left\{ \int_0^1 dx\, \hat{\rho}\frac{\partial \rho}{\partial t} - H \right\};$$

$$H = \int_0^1 dx \left\{ \frac{\sigma(\rho)}{2}(\partial_x \hat{\rho})^2 - D(\rho)\partial_x \rho\, \partial_x \hat{\rho} \right\}$$

Use the change of variable to solve least-action path

$$\hat{\rho}(x,t) = \int_{F(x,t)}^{\rho(x,t)} \mathrm{d}z \left[\frac{2D(z)}{\sigma(z)}\right]$$

Non-interacting	SEP	KMP	SIP	ZRP	RAP	GL
$D=1$	$D=1$	$D=1$	$D=1$	$D=g'(\rho)$	$D=a\rho^{-2}$	$D=D(\rho)$
$\sigma=2\rho$	$\sigma=2\rho(1-\rho)$	$\sigma=2\rho^2$	$\sigma=2\rho(1+\rho)$	$\sigma=2g(\rho)$	$\sigma=b\rho^{-1}$	$\sigma=1$

Work in progress for more general case…

For density: optimal profile evolves with time!

Hamilton's Equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[D(\rho)\frac{\partial \rho}{\partial x}\right] - \frac{\partial}{\partial x}\left[\sigma(\rho)\frac{\partial \hat{\rho}}{\partial x}\right]$$

$$\frac{\partial \hat{\rho}}{\partial t} = -D(\rho)\frac{\partial^2 \hat{\rho}}{\partial x^2} - \frac{\sigma'(\rho)}{2}\left(\frac{\partial \hat{\rho}}{\partial x}\right)^2$$

Boundary condition

$$\rho(x,-\infty) = \bar{\rho}(x); \qquad \rho(x,0) = r(x)$$

$$\rho(0,t) = \rho_a; \qquad \rho(1,t) = \rho_b$$

$$\hat{\rho}(0,t) = 0; \qquad \hat{\rho}(1,t) = 0$$

A solution

Make a local transformation

$$\widehat{\rho}(x,t) = \int_{F(x,t)}^{\rho(x,t)} \mathrm{d}z \left[\frac{2D(z)}{\sigma(z)}\right]$$

The two least-Action equations become

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[D(\rho)\frac{\partial \rho}{\partial x}\right] = 2\frac{\partial}{\partial x}\left[\frac{\sigma(\rho)D(F)}{\sigma(F)}\frac{\partial F}{\partial x}\right]$$

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x}\left[D(F)\frac{\partial F}{\partial x}\right] = \left[1 + \frac{D(\rho)}{D(F)}\right]\frac{\partial}{\partial x}\left[D(F)\frac{\partial F}{\partial x}\right] - \frac{D(\rho)\sigma'(F) - \sigma'(\rho)D(F)}{\sigma(F)}\left(\frac{\partial F}{\partial x}\right)^2$$

A particular solution

$$\frac{\partial F}{\partial t} + \frac{\partial}{\partial x}\left[D(F)\frac{\partial F}{\partial x}\right] = 0$$

$$\left[1 + \frac{D(\rho)}{D(F)}\right]\frac{\partial}{\partial x}\left[D(F)\frac{\partial F}{\partial x}\right] - \frac{D(\rho)\sigma'(F) - \sigma'(\rho)D(F)}{\sigma(F)}\left(\frac{\partial F}{\partial x}\right)^2 = 0$$

at all times t

Is this consistent with boundary condition? **Yes**

The boundary conditions now translate on F(x,t)

$$F(0,t) = \rho_a \qquad F(1,t) = \rho_b$$

$$F(x,-\infty) = \bar{\rho}(x)$$
$$F(x,0) = g(\rho(x,0))$$

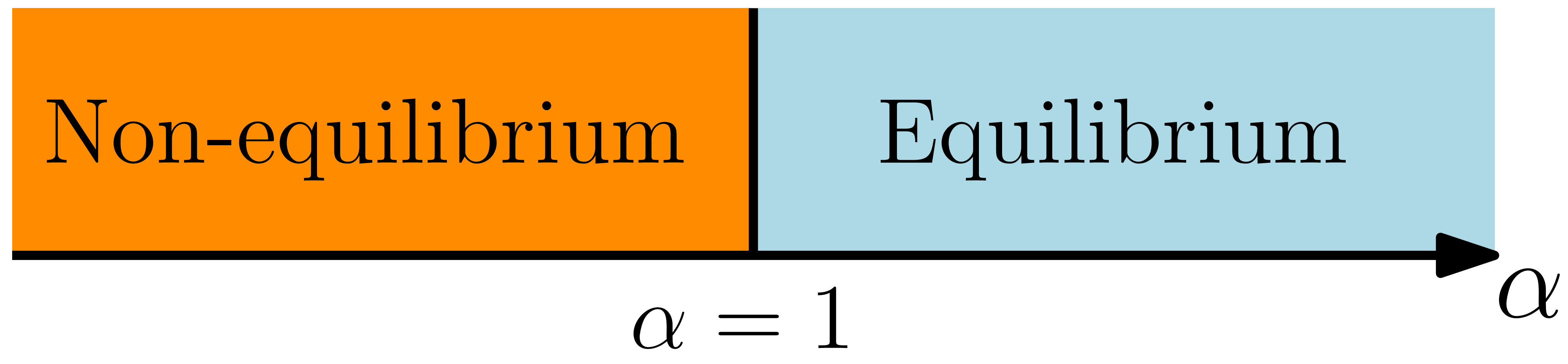
Works for a limited set of models. For more general cases we are working on a solution using perturbation theory ….

Large deviations: rest is algebra

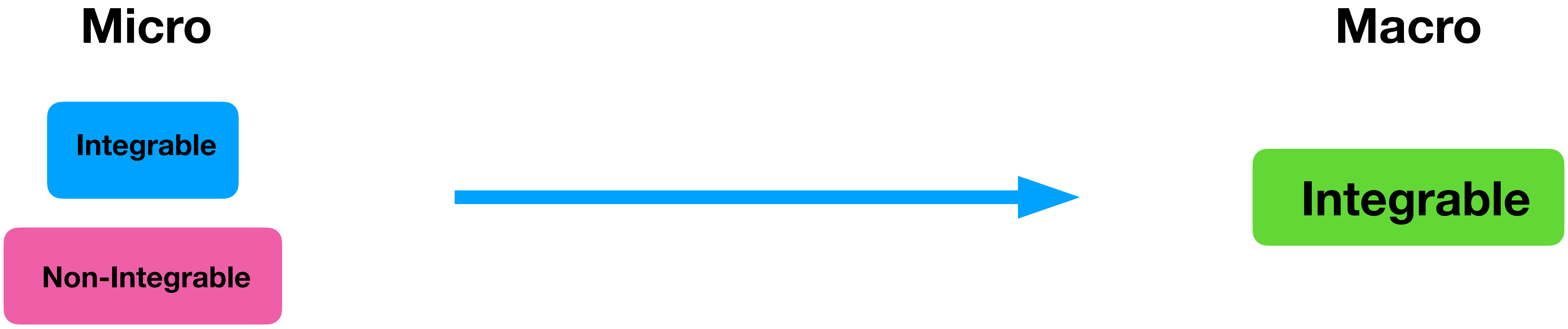
Take-home message

Summary

Robustness of LDF



“Universality” for non-equilibrium



Thank you!