On the derivation of new non-classical hydrodynamic equations for a Boltzmann gas and for Hamiltonian particle systems

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Outline

Hydrodynamical limit for the Boltzmann equation and particle systems

- Hydrodynamic limits
- Boltzmann equation
- New equations
- Result in the stationary case
- Particle systems
- New equations for particles
- Stochastic models

Motivations

The problem of deriving the hydrodynamical equations from the Hamiltonian equations of motion of atoms, is one of the main open problems of non-equilibrium Statistical Mechanics.

"Sixth Hilbert problem"

There is a scale separation between the microscopic and hydrodynamic description of a system of particles. The non-dimensional scale parameter ε is defined as the ratio between the characteristic microscopic lenght and the macroscopic one.

The hydrodynamic eqs have to be derived in the limit when the scale parameter ε goes to zero.

This program has not been completed:

Results

The argument is formal. All the results on the hydrodynamic limit for particle systems are formal: need of ergodic and mixing properties for the Hamiltonian dynamics.

Formal argument for getting:

- Euler *Morrey* (1955),

De Masi, Ianiro, Pellegrinotti, Presutti (1984)
based on the hypotesis of validity of local equilibrium

Rigorous results for stochastic models

- Euler: *Olla, Varadhan, Yau* (1991) for the Newton system

perturbed with suitable stochastic forces.

The situation is more involved for the dissipative equations

The compressible Navier-Stokes system (CNSE) is a phenomenological description of the dissipative hydrodynamics and the incompressible Navies-Stokes-Fourier system (INSF) can be derived from the compressible one in the low Mach number limit.

The CNSE has no scaling space-time invariance and hence cannot be obtained from a microscopic description while the INSF system can

Formal argument for getting:

- Incompressible Navier-Stokes-Fourier

Esposito-Marra 1993

This method gives also the form of the transport coefficients, in terms of non hydrodynamic modes in the theory.

Rigorous results for stochastic system of particles on the lattice

- Incompressible Navier-Stokes: *Esposito, Marra, Yau* 1996 for a model of particles on the lattice (Araujio, Goncalves, Simas 2023)

Formulation of the problem.

Many more rigorous results are available starting from the Boltzmann equation.

Rarefied gas in a container Ω described by the $F(\xi, \mathrm{v}, \tau) \geq 0$ the probability density on the phase space $\Omega \times \mathbb{R}^3$ solution of Boltzmann equation

$$\boxed{\frac{\partial F}{\partial \tau} + v \cdot \nabla_\xi F = Q(F,F)}$$

$$Q(f,g)(v) = \frac{1}{2}\int_{\mathbb{R}^3} dv_* \int_{\{|\omega|=1\}} d\omega B(\omega, v - v_*) \times \left\{f(v')g(v'_*) - f(v)g(v_*)\right\}$$

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$$v' = v - \omega[\omega \cdot (v - v_*)], \quad v'_* = v_* + \omega[\omega \cdot (v - v_*)].$$

B cross section. Hard spheres: $B(\omega, v - v_*) = |(v - v_*) \cdot \omega|$.

Knudsen number - Rescaled Boltzmann Equation

Solutions close to hydrodynamics

The Knudsen number ε is the mean free path in macroscopic units. After rescaling space and time to macroscopic units, (diffusive scaling: $\xi = \varepsilon^{-1}x, \tau = \varepsilon^{-2}t$)

$F^{\varepsilon}(x, v, t) = F(\varepsilon^{-1}x, v, \varepsilon^{-2}t)$

$$\frac{\partial F^{\varepsilon}}{\partial t} + \frac{1}{\varepsilon} v \cdot \nabla_x F^{\varepsilon} = \frac{1}{\varepsilon^2} Q(F^{\varepsilon}, F^{\varepsilon})$$

Initial data

Low Mach numbers regime

$$F^\varepsilon = \mu + \varepsilon\sqrt{\mu} f^\varepsilon$$

μ global Maxwellian

$$\mu(v) := \rho\big(2\pi T\big)^{-3/2} \exp\left(-\frac{|v|^2}{2T}\right)$$

Initial data : global Maxwellian $+\,\varepsilon$ terms

$$\lim_{\varepsilon\to 0} f^\varepsilon = ?$$

Incompressible Navier-Stokes-Fourier system

Formally, if $f^\varepsilon \to f_0$ in a suitable (weak) sense, then

$$f_0 = \sqrt{\mu}[\rho + u \cdot v + \frac{1}{2}\theta(|v|^2 - 3)]$$

with ρ, u, θ solving

$$\begin{aligned}
&\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0,\\
&\frac{\partial u}{\partial t} + u \cdot \nabla_x u + \nabla_x p = \lambda \Delta u\\
&\frac{\partial \theta}{\partial t} + u \cdot \nabla_x \theta = \kappa \Delta \theta,
\end{aligned}$$

where p is the pressure and $\lambda > 0$ and $\kappa > 0$ the transport coefficients.

Incompressible Navier-Stokes-Fourier system

Many results

- Hilbert expansion
  De Masi-Esposito-Lebowitz 1989
  Esposito-Guo-Kim-Marra 2018-2020
- Entropy method
  Bardos-Golse-Levermore 1991
  Golse-Saint-Raymond 2004-2009

Partial results for particular geometries

- R.Esposito, J.L.Lebowitz and R. Marra, JSP 1998-Boussinesq
- L. Arkeryd, R. Esposito, R.Marra and A. Nouri
  AS 2008, ARMA 2010, KRM 2012- Railegh Benard

  Usually, Hilbert expansion method

Different initial conditions

Consider instead initial conditions for $F^{\varepsilon}(x,v,t)$ with gradient of density and the temperature of order 1 (not perturbation of order of ε) as before, while the velocity field is still of order of the Knudsen number ε (low Mach number regime).

$$F^{\varepsilon}(x,v,0) = \mu(x,v) + \varepsilon f^{\varepsilon}$$

$$\mu(x,v) := \frac{\rho(x)}{\big(2\pi T(x)\big)^{3/2}} \exp\left(- \frac{|v|^2}{2T(x)} \right)$$

$$\rho^{\varepsilon} = \rho + \varepsilon\bar{\rho};\ T^{\varepsilon} = T + \varepsilon\bar{T};\ u^{\varepsilon} = \varepsilon u$$

Suppose Ω be a torus.

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The formal limit is different: Burnett-like terms are present
De Masi-Esposito-Lebowitz 1989, Bobylev 1995, Sone's books

$$\partial_t \rho + \nabla \cdot (\rho u) = 0; \quad \nabla(\rho T) = 0$$
$$\frac{5}{2}\rho T \nabla \cdot u = \frac{2}{3}\nabla \cdot (\kappa \nabla T),$$
$$\rho(\partial_t + u \cdot \nabla)u + \partial p = \nabla_x \cdot (\tau^{(1)} + \tau^{(2)}),$$

$$\tau_{ij}^{(1)} := \lambda \left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_i u_i \right)$$

$$\tau_{ij}^{(2)} := -\frac{\lambda^2}{P}\left[K_1 \left(\partial_i \partial_j T\right) + \frac{K_2}{T}\left(\partial_i T\right)\left(\partial_j T\right) \right]^{Tr}$$

With initial condition with gradient of order ε the perturbation is too small to generate the new terms in the limiting equations.

These new physically relevant eqs are non-classical in the sense that cannot been derived from the compressible Navier-Stokes equations. We could say that in some sense they indicate a failure of the CNS eqs in describing the real word

These equations can explain for example
• the thermal creep phenomenon: velocity flux on the surface in presence of a gradient of temperature
• the thermal convection in absence of gravity

Experiments Recent review: *H. Akhlaghi, E. Roohi, S. Stefanov , Physics Reports 2023*

Problem of interest and investigated in kinetic theory:
All the papers by the Kyoto group
C. Bardos, C.D. Levermore, S. Ukai, Tong Yang 2008 (Ac Sinica)
Well posedness of eqs.:
C.D. Levermore, W. Sun, K Trivisa 2012 SIAM
F. Huanfg W. Tan 2017 (Siam)

Rigorous results:

- Stationary problem

Only results in one dimension (and stationary)
Esposito- Lebowitz-Marra CMP1994, JSP1995,
Brull, KRM 2008,
Arkeryd-Esposito- Marra-Nouri KRM 2011

New result in general stationary situations in bounded domain
R. Esposito, Y. Guo, Lei Wu, RM preprints 2023

Stationary Boltzmann equation

$$\boxed{v \cdot \nabla_x F = Q(F, F)}$$

Boundary conditions have to be prescribed on the *incoming set*, $\gamma_- = \{(x, v) \in \partial\Omega \times \mathbb{R}^3 : n(x) \cdot v < 0\}$.

Stationary

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- **Diffuse reflection**: thermal wall:
  Given $T_w(x)$ on $\partial\Omega$, for $(x, v) \in \gamma_-$,

  $$\boxed{F(x, v) = \mathscr{P}_w F(x, v).}$$

  where

  $$\mathscr{P}_w F(x, v) = M_{T_w(x)}(v) \int_{v' \cdot n(x) > 0} v' \cdot n(x) F(x, v', t) dv',$$

  $$M_T = \frac{1}{2\pi T^2} \exp[-\frac{|v|^2}{2T}], \quad \int_{v \cdot n > 0} dv |v \cdot n| M_T(v) = 1$$

  Mass flux on $\partial\Omega$ vanishes.

Stationary

Rescale space: $F^\varepsilon(x,v) = F(\varepsilon^{-1}x, v)$

$$v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon)$$

If the gradient of temperature on the boundary are of order ε :

$$T_w = \bar{T} + \varepsilon\tau$$

the limiting equations are the stationary incompressible Navier-Stokes-Fourier

Esposito, Guo, Kim, RM,2018 construction of a solution of the form

$$F^\varepsilon = \mu + \varepsilon F_1$$

μ global Maxwellian. $\nabla T^\varepsilon, \nabla \rho^\varepsilon, u^\varepsilon = O(\varepsilon)$ (low Mach number limit)

Ghost effect

Instead require temperature $T = \bar{T} + \delta\tau$ (δ independent of ε) on the boundary and u of order ε.

Stationary Boltzmann equation in a domain $\Omega \in R^3$ with smooth general boundary, with diffusive B.C. with a profile of temperature $T_w(x)$ with gradient of order 1

We construct a solution in the form

$$F^\varepsilon = \mu + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon f_1^B + \varepsilon \mathscr{R}$$

μ local Maxwellian ρ and T functions of x

$$\mu(x,v,t) := \frac{\rho(x)}{\big(2\pi T(x)\big)^{3/2}} \exp\left(-\frac{|v|^2}{2T(x)}\right)$$

$$F_1 = -\mu^{\frac{1}{2}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} + \mu\left(\frac{\rho_1}{\rho} + \frac{u(x)\cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right)$$

$$\bar{\mathscr{A}} := v \cdot \left(|v|^2 - 5T \right) \mu^{\frac{1}{2}} \in \mathbb{R}^3, \quad \mathscr{A} := L^{-1} \bar{\mathscr{A}} \in \mathbb{R}^3,$$

$$Lf = -\frac{1}{\sqrt{\mu}} [Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)]$$

f_1^B is a boundary layer term solution of a suitable Milne problem

In this case the limiting equation are not the Navier-Stokes equation but the so called stationary ghost effect equations
for

$$\rho(x), T(x), u(x)$$

Equation

$$\nabla_x P = \nabla_x(\rho T) = 0, \quad \nabla_x \cdot (\rho u) = 0,$$

$$\frac{P}{T} u \cdot \nabla_x u + \nabla_x \mathfrak{p} = \nabla_x \cdot (\tau^{(1)} + \tau^{(2)}),$$

$$\nabla_x \cdot \left(\kappa \frac{\nabla_x T}{2T^2} \right) = 5P(\nabla_x \cdot u)$$

with the boundary conditions

$$T\Big|_{\partial\Omega} = T_w, \quad u\Big|_{\partial\Omega} := (u_{\iota_1}, u_{\iota_2}, u_n)\Big|_{\partial\Omega} = \Big(\beta_0 \partial_{\iota_1} T_w, \beta_0 \partial_{\iota_2} T_w, 0\Big).$$

(ι_1, ι_2, n) local variables $\beta_0 = \beta_0[T_w]$ is a function of T_w,

$$\tau_{ij}^{(1)} := \lambda \left(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_i u_i \right)$$

$$\tau_{ij}^{(2)} := -\frac{\lambda^2}{P} \Big[K_1 \big(\partial_i \partial_j T\big) + \frac{K_2}{T} \big(\partial_i T\big) \big(\partial_j T\big) \Big]^{Tr}$$

Transport

Explicit expressions for transport coefficients
$k(T) > 0$, thermal conductivity
$\lambda(T) > 0$ viscosity
constants K_1, K_2

$$\kappa \mathbf{1} := \int_{\mathbb{R}^3} \left(\mathscr{A} \otimes \bar{\mathscr{A}} \right) \mathrm{d}v, \qquad \lambda := \frac{1}{T} \int_{\mathbb{R}^3} \mathscr{B}_{ij} \bar{\mathscr{B}}_{ij} \text{ for } i \neq j$$

$$\bar{\mathscr{B}} = \left(v \otimes v - \frac{|v|^2}{3} \mathbf{1} \right) \mu^{\frac{1}{2}} \in \mathbb{R}^{3\times 3} \quad \mathscr{B} = L^{-1} \bar{\mathscr{B}} \in \mathbb{R}^{3\times 3},$$

$$\bar{\mathscr{A}} := v \cdot \left(|v|^2 - 5T \right) \mu^{\frac{1}{2}} \in \mathbb{R}^3, \quad \mathscr{A} := L^{-1} \bar{\mathscr{A}} \in \mathbb{R}^3,$$

Green-Kubo

"Green-Kubo formulas" for viscosity and conductivity

$$\lambda = -\frac{1}{T}\int_0^\infty ds < (|v_i v_j - \frac{1}{3}|v|^2 I)(s)(|v_i v_j - \frac{1}{3}|v|^2 I)(0) >_\mu \quad (1)$$

$$k = -\int_0^\infty ds < \frac{1}{2}(|v|^2 - 5T)v_i(s)\frac{1}{2}(|v|^2 - 5T)v_i(0) >_\mu \quad (2)$$

New transport coefficients

$$\frac{\lambda^2}{P}K_1 = \frac{1}{T^2}\int dv \mathscr{B}_{ij} v_i \mathscr{A}_j, \quad i \neq j$$

$$= \int_0^\infty ds \int_s^\infty d\tau < \bar{\mathscr{B}}_{ij} v_i(s) \bar{\mathscr{A}_j}(\tau - s >_\mu$$

Usual approach: To get the energy inequality take the scalar product with $R = \mathbf{P}R + (\mathbf{I} - \mathbf{P})R$

$\mathbf{P}$ projector on the null space

$$\varepsilon^{-1} \left|(1 - \mathscr{P})R\right|^2_{L^2_{\gamma_+}} + \varepsilon^{-2} \left\|(\mathbf{I} - \mathbf{P})R\right\|^2_{L^2_\nu}$$
$$\lesssim -\varepsilon^{-1} \iint_{\Omega \times \mathbb{R}^3} \left(\mu^{-\frac{1}{2}} \bar{\mathscr{A}} \cdot \frac{\nabla_x T}{4T^2}\right) R^2 + o(1) \left|\mathscr{P}[R]\right|^2_{L^2_\gamma} + \varepsilon^{-1} \langle R, S\rangle$$

Main difficulty: diverging term in the energy estimate for the remainder.

Solution: smart way of canceling the bad terms

New decomposition of R:

$$R = \mathbf{P}R + \mathscr{A} \cdot \mathbf{d} + (\mathbf{I} - \bar{\mathbf{P}})R$$

More tools

- boundary layer terms
- new estimate for Milne problem
- positivity
- well-posedness
- L^6 estimates for the strong nonlinearity
- L^∞ estimates

Results: *R. Esposito, Y. Guo, Lei Wu, RM preprints 2023*

Working space

norm:

$$|||R||| : \|R\|_{L^2} + \varepsilon^{\frac{1}{2}} ||R||_{L^\infty}$$

$$\begin{aligned}\|R\|_X := &\varepsilon^{-1}\|p\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\mathbf{b}\|_{L^2} + \|c\|_{L^2} + \varepsilon^{-1}\|\xi\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\xi\|_{H^2} \\ &+\varepsilon^{-1}\|\mathbf{e}\|_{L^2} + \varepsilon^{-1}\left\|(\mathbf{I} - \bar{\mathbf{P}})[R]\right\|_{L^2_\nu} + \|p\|_{L^6} + \|\mathbf{b}\|_{L^6} + \|c\|_{L^6} \\ &+\varepsilon^{-1}\|\xi\|_{L^6} + \|\xi\|_{W^{2,6}} + \|\mathbf{e}\|_{L^6} + \left\|(\mathbf{I} - \bar{\mathbf{P}})[R]\right\|_{L^6} + |\mathscr{P}[R]|_{L^2_\gamma} \\ &+ \varepsilon^{-\frac{1}{2}}|(1-\mathscr{P})[R]|_{L^2_{\gamma_+}} + \left|\mu^{\frac{1}{4}}(1-\mathscr{P})[R]\right|_{L^4_{\gamma_+}} + \varepsilon^{-\frac{1}{2}}|\nabla_x\xi|_{L^2_{\partial\Omega}} \\ &+ \varepsilon^{\frac{1}{2}}\|R_M\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}}|R_M|_{L^\infty_{\gamma,\varrho,\vartheta}}.\end{aligned}$$

Result

Theorem:
Assume that Ω is a bounded C^3 domain. Assume $|\nabla T_w|_{W_{3,1}}$ small. Then for any given $P > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a nonnegative solution $F^\varepsilon = \mu + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon f_1^B + \varepsilon R$ to the equations satisfying $\int_\Omega p(x)\mathrm{d}x = 0$

$$|||R||| \lesssim o_T$$

with o_T a small constant such that $o_T \to 0$ as $|\nabla T_w|_{W_{3,1}} \to 0$

Remark

In particular, for

$$R := \mu^{\frac{1}{2}}(v)\left(p(x) + v \cdot \boldsymbol{b}(x) + \left(|v|^2 - 5T\right)c(x)\right) + \mathscr{A} \cdot \boldsymbol{d} + (\boldsymbol{I} - \bar{\boldsymbol{P}})R$$

$$\|p\|_{L^2} \lesssim o_T\varepsilon,\ \|\boldsymbol{b}\|_{L^2} \lesssim o_T\sqrt{\varepsilon},\ \|d\|_{L^2} \lesssim o_T\varepsilon,\ \|c\|_{L^2} \lesssim o_T$$
$$\left\|(\boldsymbol{I} - \bar{\boldsymbol{P}})[R]\right\|_{L^2_\nu} \lesssim o_T\varepsilon$$

Hamiltonian particle systems

Hamiltonian system of particles interacting by means of a smooth pair potential $V(|\xi_i - \xi_j|$.
The evolution is described by the Newton equations. Rescale space as ε and time as ε^2.
Macroscopic variables $x_i = \varepsilon \xi_i, t = \varepsilon^2 \tau$

$$\begin{aligned} \frac{dx_i}{dt}(t) &= \varepsilon^{-1} v_i(t), \\ \frac{dv_i}{dt}(t) &= -\varepsilon^{-2} \sum_{i \neq j} \nabla V(\varepsilon^{-1}|x_i - x_j|) \end{aligned}$$

The number of particles N is assumed to be of order ε^{-d}

Local conserved quantities

empirical density

$$z^0(x) = \varepsilon^d \sum_{i=1}^{N} \delta(x_i - x)$$

empirical velocity field density

$$z^\alpha(x) = \varepsilon^d \sum_{i=1}^{N} v_i^\alpha \delta(x_i - x), \quad \mathscr{A} = 1, \ldots, d$$

empirical energy density

$$z^{d+1}(x) = \varepsilon^d \sum_{i=1}^{N} \frac{1}{2}\big[v_i^2 + \sum_{j \neq i = 1}^{N} V(\varepsilon^{-1}|x_i - x_j|)\big]\delta(x_i - x)]$$

The average of the integral of z^α over a small region is equal to the average number of particles, momentum, energy associated to the region (candidates for the hydrodynamical fields in the limit).

Local conservation laws in terms of currents$w^{\beta k}(x)$

$$\frac{\partial}{\partial t} \int dx f(x) z^{\beta}(x) = \varepsilon^{-1} \int dx \sum_{k=1}^{d} \partial_k f(x) w^{\beta k}(x)$$

The currents are not functions of the empirical fields. We need a closure argument to get the hydrodynamic equations.

We take the average wrt the Liouville measure

Take the average versus F_ε solutions of the Liouville equations and assume a particular form in terms of an expansion in ε. Analogous to the Hilbert expansion.

In the limit we get the hydrodynamic eqs.

Initial condition local Gibbs measure + order ε corrections

$$G(\underline{\lambda}) = Z_\Lambda^{-1} \exp\left[\int_{\varepsilon^{-1}T^d} d\xi \sum_{\alpha=0,d+1} \lambda^\alpha(x) z^\alpha(\xi)\right]$$

λ chemical potentials functions of macroscopic variables

$$\lambda^0 = \log\tilde{z} - \frac{1}{2}\beta|u|^2, \quad \lambda^\nu = \frac{1}{T}u^\nu, \nu = 1, \cdots, d \quad \lambda^{d+1} = -\frac{1}{T}.$$

$\lambda^\nu = 0$ low Mach number assumption.
Gradient of density and temperature of order 1

–The local Gibbs measure is the analogous of the local Maxwellian–

Rescaled Liouville equation

$$\frac{\partial F_\varepsilon}{\partial t} = \varepsilon^{-2}\mathscr{L}^* F_\varepsilon.$$

Write F_ε as $F_\varepsilon = G_\varepsilon + \varepsilon G_0 R_\varepsilon$

$$G_\varepsilon = Z_\varepsilon^{-1}\exp\{\sum_{i,\mu}\lambda_\varepsilon^\mu(x_i,t)z_i^\mu\},$$

$$\lambda_\varepsilon^\mu(x,t) = \sum_{n=0}^{\infty}\varepsilon^n\lambda_n^\mu(x,t); \qquad \lambda_0^0(x,t)\neq 0, \lambda_0^{d+1}(x,t)\neq 0,$$

$$\lambda_0^\mu = 0,\ \mu = 1,\ldots,d;$$

R_ε non-hydrodynamic modes

R. Esposito, RM, preprint 2023: In the formal limit $\varepsilon \to 0$

$$\begin{cases} \nabla P &= 0, \\ \rho\partial_t u + u\cdot\nabla_x u + \nabla_x \mathfrak{p} &= \nabla_x\cdot\left(\tau^{(1)} - \tau^{(2)}\right), \\ \partial_t\rho + \nabla_x\cdot(\rho u) &= 0, \\ \rho[\partial_t e + u\cdot\nabla e] + P\big(\nabla_x\cdot u\big) &= \nabla_x\cdot\left(\kappa\dfrac{\nabla_x T}{2T^2}\right), \end{cases}$$

$\alpha, \beta = 1, \dots d$, with d the space dimension,

$$\tau^{(1)}_{\alpha\beta} := \eta\left(\partial_\alpha u^\beta + \partial_\beta u^\alpha - \frac{2}{d}\delta_{\alpha\beta}\partial_\alpha u^\alpha\right) + \zeta\delta_{\alpha\beta}\partial_\alpha u^\alpha,$$

$$\tau^{(2)}_{\alpha\beta} := -[K_1(\partial_\alpha T\partial_\beta T - \frac{1}{d}\sum_\alpha(\partial_\alpha T)^2) + \omega_1\sum_\alpha(\partial_\alpha T)^2$$

$$+K_2(\partial^2_{\alpha\beta}T - \frac{1}{d}\sum_\alpha\partial^2_{\alpha\alpha}T) + \omega_2\sum_\alpha\partial^2_{\alpha\alpha}T)]$$

Comments

- P and e are the thermodynamical potentials pressure and internal energy which are functions of ρ, T as determined by the Gibbs measure.
- the equations reduce the ones for Boltzmann for a perfect gas
- Transport coefficients
  viscosity coefficient η, bulk viscosity ζ and the conductivity κ given by the Green-Kubo formulas
- K_1, K_2 new transport coefficients not given by the linear response theory (double time integral of correlation of some currents).
- entropy associated to the solutions of these equations is increasing in time.

Stochastic model

Possible stochastic model for these new equations:
"Navier-Stokes limit for a thermal stochastic lattice gas"
Benois, Esposito, Marra, 1999

Particles jumping on the lattice with velocity and kinetic energy.
Result : Deriving Incompressible Navier-Stokes-Fourier eqs

Description of the model:

Particles of a finite number of colors (velocities), labeled by vectors $v \in \mathbb{R}^3$. The particles of the color v move on the lattice $\mathbb{Z}^3$ according to the simple exclusion process, i.e. they can only jump, at independent exponential times, to nearest neighbors (provided that the target site in not already occupied by a particle of the same color) with rates such that the drift is v.
Two particles collide when they are in the same site and have suitable velocities. The essential condition on the collisions is that they conserve the number of particles involved in the collision, their total momentum and energy.

Different energies: two values of $|v|$: $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$ with $\mathscr{V}_1 = \{(\pm 1, \pm 1, \pm 1)\}$ and $\mathscr{V}_2 = \{(\pm 1, \pm 1 \pm \varpi)\}$, up to permutations, with $\varpi \neq 1$.

Since the energy is non trivial, a heat equation is also obtained in the diffusive scaling limit.

Assume $d = 3$ and the initial distribution of the system to be of local equilibrium with chemical potentials satisfying the low Mach numbers assumption (gradient of order ε).

$$\nabla \cdot u = 0 \tag{3}$$

$$\partial_t u_\beta + \nabla p + K\, u \cdot \nabla u_\beta = \sum_{\alpha=1}^{3} D_{\alpha,\beta} \partial_\alpha^2 u_\beta \tag{4}$$

$$\frac{\partial}{\partial t} T + H u \cdot \nabla T = \sum_{\alpha=1}^{3} \mathscr{K}_\alpha \partial_\alpha^2 T. \tag{5}$$

Different initial condition: gradient of order 1 for density and temperature in the local Gibbs measure

- which kind of equations?
- Varadhan non-gradient method will be enough?