IHP 23/01/2024

Kinetic Theory of Weakly Nonlinear Wave Equations

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- wave turbulence — weakly nonlinear wave equations
  random initial data, extensive
- precursor: Peierls (1930) — anharmonic crystals
  thermal conductivity
- nonlinear Schrödinger equation NLS — top favorite
- quantized lattice fermions — in progress

Lukkarinen, HS
"Not to normal order" (2009)

NLS: wave field $\psi : [-\ell, \ell]^d \to \mathbb{C}$, $d \geq 3$, periodix box

$$i \partial_t \psi = - \Delta \psi + \lambda |\psi|^2 \psi$$

on-site

smeared $|\psi|^2 * \varphi(x)\, \psi(x)$

- initial data Gaussian U(1) invariant

$$\langle \psi(x) \rangle = 0 \,, \quad \langle \overline{\psi}(x) \psi(y) \rangle = w(x-y) \,, \quad \langle \psi \psi \rangle = 0 = \langle \overline{\psi}\, \overline{\psi} \rangle$$

$$\hat{w}(k) = W^0(k) \,, \quad k \in \frac{1}{2\ell} \mathbb{Z}^d$$

averaged Wigner function $\hat{=}$ momentum density

semiclassical limit

$\Rightarrow$ $\lambda = 0$ $\quad \langle \cdot \rangle$ is stationary

$\Rightarrow$ small λ

$$\langle \overline{\psi}(x) \psi(y) \rangle_{t,\lambda} = w_\lambda(x-y, t) \rightsquigarrow W_\lambda(k, t) \quad ??$$

‖ spatially homogeneous ‖

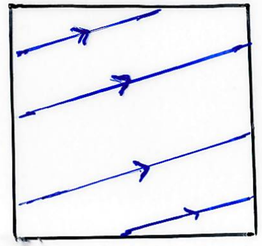
scales

- time: $t = \lambda^{-2}\tau, \quad \tau = O(1)$

- space: mean free path $\ell = \lambda^{-2}$ ballistic

$\ell = \lambda^{-\gamma}, \quad \gamma > 0$

many recollisions



Conjecture:

$d \geq 3, \quad \gamma \geq 2,$

$$\lim_{\lambda \to 0} W_\lambda(k, \lambda^{-2}\tau) = W(k,\tau)$$

$k \in \mathbb{R}^d$

lattice spacing λ^2

$W(k,0) = W^0(k)$

AND

$$\underbrace{}_{}\quad \text{momentum} \qquad\qquad \text{energy}$$

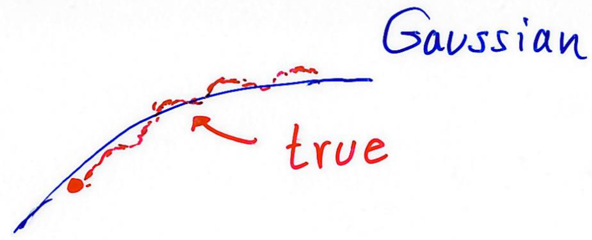
$$\partial_\tau W(k_1) = \int dk_2\, dk_3\, dk_4 \; \delta(k_1 + k_2 - k_3 - k_4)\; \delta(k_1^2 + k_2^2 - k_3^2 - k_4^2)\; |\hat{\varphi}(k_2 - k_3)|^2$$

$$\times\; W_1 W_2 W_3 W_4 \left(\frac{1}{W_1} + \frac{1}{W_2} - \frac{1}{W_3} - \frac{1}{W_4} \right) \qquad W_j = W(k_j)$$

$$= \mathcal{C}(W)(k_1)$$

physics: 2^{nd} order $W^{(2)}(k, \lambda^{-2}\tau)$, $\lim_{\lambda \to 0} W^{(2)} = \tau\, \mathcal{C}(W)$

assume: Gaussian at time $\lambda^{-2}\tau$



H - theorem

$$\lim_{\tau \to \infty} W(k, \tau) = W_{eq}(k)$$

$$W_{eq}(k) = \frac{1}{\beta(k^2 - \mu) + u k} \qquad \text{UV divergent}$$

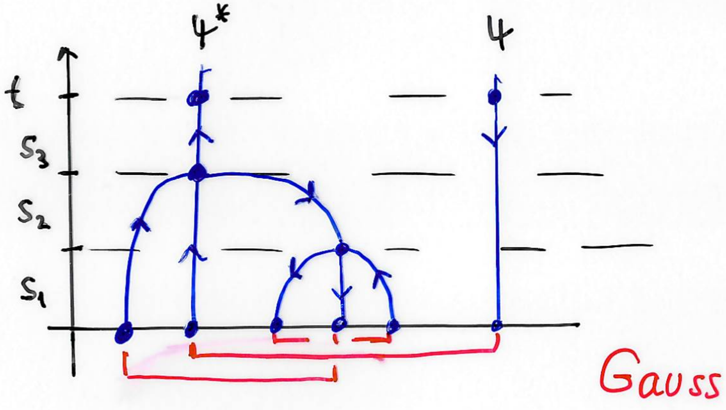
β, μ, u from initial conditions

Fourier space

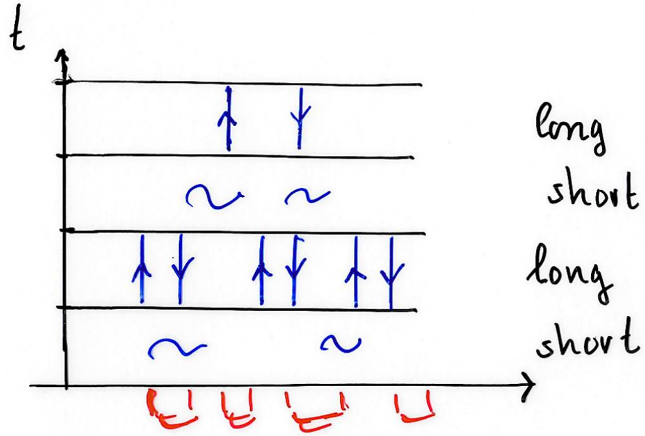
$$\partial_t \hat{\psi}(k) = -i k^2 \hat{\psi}(k) - i\lambda \int dk_1 dk_2 dk_3 \, \delta(k+k_1-k_2-k_3) \, \hat{\psi}(k_1) \hat{\psi}^*(k_2) \hat{\psi}(k_3)$$

Duhamel expansion

2^{nd} order



QM : initial state is quasi free



- leading diagrams

$\lim \lambda \to 0$

$$(\underset{\overset{\shortparallel}{\tau}}{\lambda^2 t})^n \frac{1}{n!} (\underbrace{\mathcal{C} \circ \quad \circ \mathcal{C}}_{n})(W)$$

power series $0 \leq \tau \leq \tau_0$ "finite kinetic time"

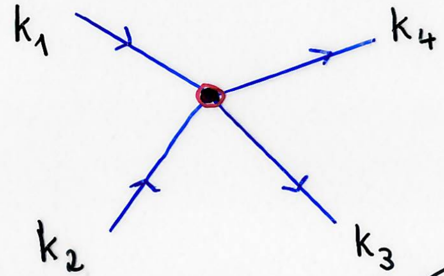
condensation of waves

Escobedo, Velázquez 2008 –

HS (2010)

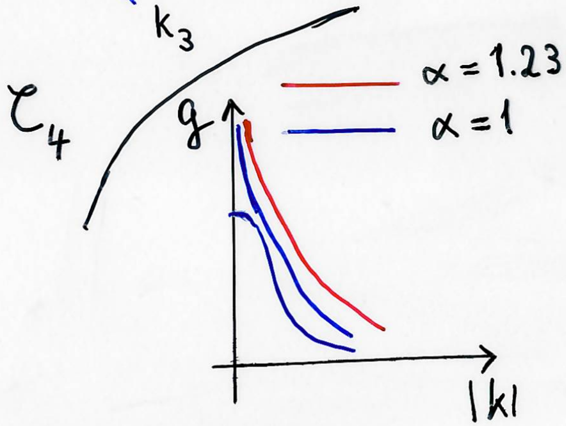
$$W(k,t) = a(t)\,\delta(k) + f(k,t)$$

condensate — normal fluid



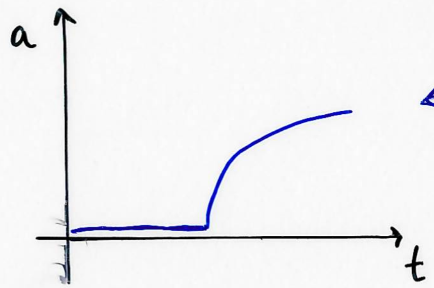
$$\rightsquigarrow \quad \partial_t f = \mathcal{C}_4(f) + a\,\mathcal{C}_3(f)$$

$$\dot{a} = -a \int \tilde{\mathcal{C}}_3(f)$$



isotropic $\quad W(k,t) = a(t)\,\delta(k) + g(|k|,t)$

$t = 0 \quad a = 0$, g supercritical



$\longleftarrow \quad g(|k|,t) = |k|^{-\alpha}$ for $|k| \to 0$, $\quad \alpha > 0$

critical

blow-up

classical kinetic equation

No blow-up

subleading diagrams

how many ? Branching $n!$, ordered time $\frac{1}{n!}(\lambda t)^n$, Gauss pairings $n!$

|| should be λ^2 || $\rightsquigarrow n!\,\lambda^n t^n$

→ strategy based on Erdös-Yau ←

$$i\partial_t \psi = -\Delta\psi + V\psi \quad , \; V(x) \text{ random Gauss}, \; \langle V(x)\rangle = 0, \; \langle V(x)V(y)\rangle = g(x-y)$$

(i) cut-off: $n = 0, 1 .., N$, $N \cong -\log\lambda$

(ii) uniform bound EY use $\|\psi(t)\| = 1$ does NOT work for NLS

(iii) high-dimensional oscillatory integrals

- renormalized $\omega(k) = k^2 \rightsquigarrow \omega_\lambda(k)$
- crossing estimate

discrete NLS

Lukkarinen, HS 2011

$$i\partial_t \psi_j = \sum_m a(j-m)\psi_m + \lambda |\psi_j|^2 \psi_j \qquad \text{on-site} \qquad j \in \mathbb{Z}^d$$

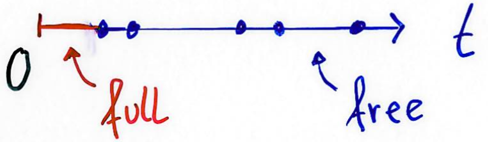
$$\hat{a} = \omega$$

- initial: equilibrium $\frac{1}{Z}\exp\Big[-\sum_{j,m} a(j-m)\bar{\psi}_j \psi_m + \lambda \sum_j |\psi_j|^4\Big]$, $\langle \cdot \rangle_\lambda$

- price ω has critical points $\neq 0$

- uniform bound $|\langle A(t) B\rangle_\lambda|^2 \le \langle |A(t)|^2\rangle_\lambda \langle |B|^2\rangle_\lambda$

$$= \langle |A(0)|^2\rangle_\lambda$$



requires Abdessalam et al (2009)

<u>Theorem</u> a n.n., $d \geq 4$, $|\tau| \le \tau_0$

$$\langle \bar{\psi}(x,t)\psi(y,0)\rangle_\lambda = g_\lambda(x-y,t)$$

$$\lim_{\lambda\to 0} e^{-i\omega_\lambda(k)\lambda^{-2}\tau}\,\hat{g}_\lambda(k,\lambda^{-2}\tau) = \hat{g}(k,\tau) \qquad \Big\| \qquad \hat{g}(k,\tau) = e^{-T(k)|\tau|}\, e^{i\tilde{T}(k)\tau}$$

loss term of linearized kinetic eq.

renewed interest 2017 – starting at Courant

Yu Deng, Zaher Hani 2019 – now 340 pages

UCS Michigan Nov 2023 + 115 pages

- NLS, spatially homogeneous, volume $\left(\frac{1}{\lambda}\right)^{\gamma}$, $\gamma > 1$
- expansion in Feynman diagrams
- $\omega(k) = k^2$ internal lines are Gaussian integrals
- magic cancellations ‖ group of diagrams "molecule"
- there is no comparison with LS

recommended ‖ Z. Hani lectures, Feb 2023 Gran Sasso Institute ‖

$\Rightarrow$ Deng, Hani arXiv 20.3.2023 // NLS // on-site //

Theorem $d \geq 3$, $1 < \gamma < \infty$, $0 \leq \tau < \tau_0$, $\ell = \left(\frac{1}{\lambda}\right)^\gamma$

(i) $\hat{\psi}_\lambda(k,t)$, $k \in \left(\frac{1}{\ell}\mathbb{Z}\right)^d$, $0 \leq t \leq \lambda^{-2}\tau_0$, initial $W^0(k)$ Schwartz

- solution exists with probability $\geq 1 - e^{-(\log \ell)^2}$

$\rightsquigarrow W_\lambda(k,t)$

kinetic eq.

(ii)

$$\lim_{\lambda \to 0} \sup_{0 \leq \tau \leq \tau_0} \sup_{k \in \left(\frac{1}{\ell}\mathbb{Z}\right)^d} \left| W_\lambda(k, \lambda^{-2}\tau) - W(k,\tau) \right| = 0$$

ALSO: law of large numbers, higher moments, non-Gaussian //

Outlook

- DNLS analogue of Deng, Hani
- Why are cancellations so effective?
- weakly coupled lattice fermions $\|a_j\| = 1$

- related activities: Thomas Chen quantum

  Amarali Hanani, Gigliola Staffilani, Binh Tran + stochastic terms