Kinetic scaling limits in plasma physics

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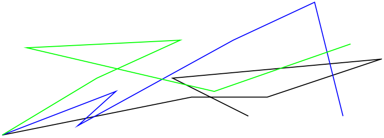
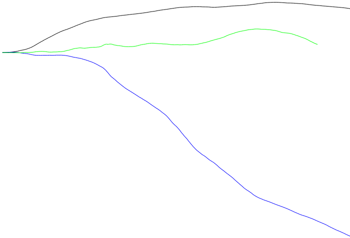
Formally: For a large class of scaling limits of interacting particle systems, limiting kinetic equations are

- (Vlasov equation:)
  $\partial_t f + v\nabla_x f + F_f \nabla_v f = 0$

- Boltzmann equation:
  $\partial_t f + v\nabla_x f = Q_B(f,f)$



- Landau equation
  $\partial_t f + v\nabla_x f = Q_L(f,f)$



- Balescu-Lenard equation
  $\partial_t f + v\nabla_x f = Q_{BL}(f,f)$

Goal: systematic classification

$n \gg 1$ particles $(X_i, V_i)_{i=1}^n \in \mathbb{T}^3 \times \mathbb{R}^3$, ϕ (short-range) potential

$$\dot{X}_i(t) = V_i(t), \quad \dot{V}_i(t) = -\sum_{i \neq j} \beta \nabla \phi_\epsilon(X_i - X_j),$$

where $n = N$ or $\mathbb{E}[n] = N$ and $\alpha \in [0,1]$

$$\phi_\epsilon(x) = \epsilon^\alpha \phi\left(\tfrac{x}{\epsilon}\right).$$

Initial data (canonical or grand-canonical type):

- symmetric, translation invariant
- chaotic
- probability distribution $f_0(v)$, $\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f_0(v) = \beta^{-1} = 1$

For simplicity:

$$(X_i, V_i) \sim Z^{-1} e^{-\sum_{i \neq j} \beta \phi_\epsilon(X_i - X_j)} \prod_{i=1}^n f_0(V_i).$$

Formal limits

For $\alpha \in [0,1]$ and

$$\dot{X}_i(t) = V_i(t), \quad \dot{V}_i(t) = -\sum_{i \neq j} \nabla \phi_\epsilon(X_i - X_j),$$

$$\phi_\epsilon(x) = \epsilon^\alpha \phi\left(\frac{x}{\epsilon}\right),$$

kinetic limit for one-particle function $f(t,v)$ under scaling

$$N = \epsilon^{-2(1+\alpha)}.$$

The limit equation is given by

- $\alpha = 0$: Boltzmann equation
- $\alpha \in (0,1)$: Landau equation $\quad \alpha = \frac{1}{2}$ 'weak-coupling' limit
- $\alpha = 1$: Balescu-Lenard equation

Systematic approach as [Spohn, Rev. Mod. Phys. 1980] and [Spohn, Springer, 1991]

State of the art: $\alpha = 0$ 'Boltzmann'

$$\partial_t f = \int_{\mathbb{R}^3} dv_1 \int_{S^2} d\sigma \, B_\phi(v - v_1, \sigma) \, \{ \, f(v')f(v_1') - f(v)f(v_1)\}$$

Known results [1]

- for ϕ hard-sphere up to $T > 0$
- for ϕ smooth, short-range up to $T > 0$

Some open problems

- global-in-time problem open
- so far no result for $\phi(x) \sim |x|^{-s}$

Related challenge: Derive (linear) Boltzmann-Vlasov equation under a scaling $N = \epsilon^{-2}$.

$$\dot{V}_i = -\sum_{i \neq j} \epsilon^{-1} \nabla \phi_1((X_i - X_j)/\epsilon) - N^{-1} \sum_{i \neq j} \nabla \phi_2(X_i - X_j).$$

[1] [Lanford], [Bodineau, Gallagher, Saint-Raymond, Simonella] [Saffirio,Pulvirenti] and more

Theorem [Lutsko, Toth, CMP, 2020]

Let X^ϵ be flight process of tagged particle in ϵ hard-sphere Lorentz gas, and

$$T(\epsilon) \to \infty, \quad \epsilon^2 T(\epsilon) |\log(\epsilon)|^2 = o(1).$$

Then for any $\delta > 0$ we have

$$P(\sup_{0 \le t \le T} |X^\epsilon(t) - Y(t)| \ge \delta \sqrt{T}) \to 0,$$

where Y is the appropriate Markovian flight process.

New coupling technique that realizes the processes X^ϵ, Y together with a 'short-sighted' process Z on a joint probability space.

Corrections for low but positive volume fraction

Studied extensively by physicists (Uhlenbeck, Cohen, Murphy, Resibois, ...):
Develop a theory for small but positive ϵ.

Goal: Mathematically rigorous formalism for correction of the Lanford result:

$$\|f_\epsilon - f\| \to 0, \quad \partial_t f = Q_B(f, f)$$

Idea: Find an (explicit) family of operators Q_{CU}^ϵ and prove that

$$\|f_\epsilon - \overline{f}_\epsilon\| \leq o(\epsilon), \quad \text{where} \quad \partial_t \overline{f}_\epsilon = Q_B(f, f) + \epsilon Q_{CU}^\epsilon(\overline{f}_\epsilon)$$

Postulates underlying the Boltzmann equation:

1. Collisions are purely binary
2. Collisions are localized in space and time
3. Particles are independent prior to collision

Operators Q_{CU}^ϵ need to take into account corrections to these postulates!

Theorem [Simonella, W. 2023, preprint]

Let $f_\epsilon(t)$ be the one-particle marginal of the hard-sphere system with distribution $\mu_{f_{1,0}}$. Let $\overline{f}_\epsilon$ be the solution to the kinetic equation:

$$\begin{aligned} \partial_t \overline{f}_\epsilon + v\nabla \overline{f}_\epsilon &= Q_B(f,f) + \epsilon Q_{CBE}(\overline{f}_\epsilon, \overline{f}_\epsilon, \overline{f}_\epsilon) \\ \overline{f}_\epsilon(0,\cdot) &= f_\epsilon(0,\cdot). \end{aligned}$$

Then for some $T > 0$ there holds:

$$\lim_{\epsilon \to 0} \epsilon^{-1} \sup_{x\in\mathbb{R}^3} \sup_{t\in[0,T]} \| f_\epsilon(t,x,\cdot) - \overline{f}_\epsilon(t,x,\cdot)\|_{L^1} = 0.$$

The Choh-Uhlenbeck operator is given by

$$Q_{CBE}(f) = \sum_{\substack{\Gamma(1,2) \\ \sigma_1,\sigma_2,\sigma_3\in\{+,-\}}} \int_{R^{\sigma_3}} \sigma_1\sigma_2\sigma_3 \prod_{i=1}^{2} [(\eta_{k_i}^{\sigma_3} - v_{i+1})\cdot\omega_i]_+ f^{\otimes 3}(\zeta_1^{\sigma_3}, \zeta_2^{\sigma_3}, \zeta_3^{\sigma_3}).$$

State of the art: $\alpha \in (0,1)$ 'Landau'

$$\partial_t f(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} B(v, v-v')(\nabla f(v) f(v') - \nabla f(v') f(v)) dv' \right)$$

$$B(v, v-v'; \nabla f) = \int_{\mathbb{R}^3} (k \otimes k) |\hat{\phi}(k)|^2 \delta\big(k \cdot (v-v')\big) dk$$

Nonlinear equation from interacting particles: –

From linear models

- from a random, mixing force field [Kesten,Papanicolaou 1980], $d \geq 3$
- from Lorentz model in $d = 2$ with short-range potential [Dürr,Goldstein,Lebowitz, 1987], mixing force field [Komorowski, Ryzhik, Isr. Jour. Math., 2006]
- so far no result for $\phi(x) \sim \langle x \rangle^{-s}$ with s small

Well-posedness of classical solutions to PDE:

- Recently achieved globally in time for spatially homogeneous case [Guillen,Silvestre, 2023]
- conditional results for spatially inhomogeneous

Other approaches to Landau limit: Boltzmann-to-Landau

Theorem [Catapano, KRM, 2018]

Consider tagged particle in heat bath, i.e.

$$W_{0,N}(x_0, v_0, z_N) = g_0(x_0, v_0) M_\beta(v_0) M_{N,\beta}(z_N),$$

and the Hamiltonian dynamics given by

$$\dot{X}_i = V_i, \quad \dot{V}_i = -\alpha^{-\frac{1}{2}} \sum_{i \neq j} \nabla \phi_\epsilon(X_i - X_j).$$

Then under the scaling

$$N\epsilon^2 = \alpha = (\log \log N)^{\frac{1}{2}}, \quad N \to \infty.$$

the density $f_{1,N}(t, x_0, v_0)$ converges to solution to linear Landau equation

$$\| f_{1,N} - g(t, x_0, v_0) M_\beta(v_0) \|_H \to 0.$$

Theorem [Le Bihan, W., KRM, 2023]

Let $\phi \in C_c^\infty(B_1)$ be a radially symmetric interaction potential with

$$\phi(x) = \frac{f(|x|)}{|x|^s}$$

for $s \geq 1$ and f monotone decreasing. Let g_ϵ be the solution to

$$\partial_t g_\epsilon = \delta_\epsilon^{-1} \mathcal{L}_\epsilon,$$

where scaling of $\mathcal{L}_\epsilon$ linearized Boltzmann operator given by

$$\phi_\epsilon(x) = \epsilon \phi(x), \qquad \delta = \begin{cases} \epsilon^{-2} & s \in [0,1) \\ \epsilon^{-2} |\log(\epsilon)| & s = 1. \end{cases}$$

Then $g_\epsilon \rightharpoonup^* g \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^6))$, where g solves linearized Landau operator associated to ϕ, s.

From truncated BBGKY hierarchy Close hierarchy at 2nd cumulant, obtain nonlinear system [Bobylev, Saffirio, Pulvirenti, CMP, 2013]

$$\partial_t f_1^\epsilon = \epsilon^{-1}\nabla_{v_1}\cdot\Big(\int_{\mathbb{R}^3}\nabla\phi(x^*)g_2^\epsilon(x_*,v_1,v_2)dx_*dv_2\Big)$$
$$\partial_t g_2^\epsilon = -(v_1-v_2)\nabla_x g_2^\epsilon + \epsilon\nabla\phi(x)\big(\nabla f_1^\epsilon(v_1)f_1^\epsilon(v_2) - f_1^\epsilon(v_1)\nabla f_1^\epsilon(v_2)\big).$$

Here $g_2^\epsilon(x_1-x_2,v_1,v_2)$ approx. of two-particle cumulant.

Question: as $\epsilon\to 0$, $f_1^\epsilon \to f$ solution to nonlinear Landau

- consistency at $t=0$ (Bobylev, Saffirio, Pulvirenti, CMP, 2013)
- for $t\in[0,T^*]$, $T^*>0$ [Velázquez, W., CMP, 2018], [W., JDE, 2021] using uniform estimates for

$$\int_0^{T^*} e^{-\lambda t}(\|f(s,\cdot)\|^2_{H^n_\omega} + D_\epsilon(f(s)))ds \le C,$$

  uniformly in $\epsilon\to 0$, and $D_\epsilon \to D$ local-in-time dissipation.

State of the art: $\alpha = 1$ 'Balescu-Lenard'

$$\partial_t f(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} B(v, v-v')(\nabla f(v) f(v') - \nabla f(v') f(v)) dv' \right)$$

$$B(v, v-v'; \nabla f) = \int_{\mathbb{R}^3} \frac{(k \otimes k)|\hat{\phi}(k)|^2}{|\epsilon(k, k\cdot v)|^2} \delta\big(k \cdot (v - v')\big) dk$$

From interacting particle systems: –

Tagged particle in heat bath [Duerinckx, Saint-Raymond]:
Consistency result on shorter timescale ($t = 0$).

Theory for the PDE:

- ϕ Coulomb [Strain, CPDE, 2006]: well-posedness of linearized equation with exponential loss of weights:

$$\|f(t,\cdot)\|_{L^2} \le e^{-\lambda t^p} \|f_0\|_{L^2_\theta}, \tag{1}$$

  where L^2_θ is exponentially weighted and $p = p(\theta)$. Problem persists for $\phi(x) = \langle x \rangle^{-s}$, $s < 3$.
- ϕ smooth short-rage [Duerinckx, W., ARMA, 2023]: Global well-posedness close to equilibrium, local away from equilibrium
- seemingly unaffected by [Guillen, Silvestre 2023]

Equation for the velocity distribution $f(v)$ of a plasma

$$\partial_t f(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} B(v, v - v')(\nabla f(v) f(v') - \nabla f(v') f(v)) dv' \right)$$

$$B(v, v - v'; \nabla f) = \int_{\mathbb{R}^3} \frac{(k \otimes k)|\hat{\phi}(k)|^2}{|\epsilon(k, k \cdot v)|^2} \delta\big(k \cdot (v - v')\big) dk$$

- **Landau equation**

$$\epsilon(k, k \cdot v) \equiv 1$$

  Used in simulations, mathematical results (Desvillettes, Guo, Mouhot, Strain, Villani, ...)

- **Balescu-Lenard equation**

$$\epsilon(k, k \cdot v) = 1 + \hat{\phi}(k) \int_{\mathbb{R}^3} \frac{k \nabla f(v_*)}{k \cdot (v - v_*) - i0} dv_*$$

Theorem [Duerinckx, W., 2021]

Let $d \geq 2$. Let $\phi \in L^1 \cap \dot{H}^2(\mathbb{R}^d)$ be isotropic and positive definite, and assume $x\phi \in L^2(\mathbb{R}^d)$. For all $s \geq 2$ and $0 < \beta < \infty$, exists $C_{V,\beta,s}$ large enough such that: for all initial data $F^\circ \in L^1(\mathbb{R}^d)$ of the form

$$F^\circ = M_\beta + \sqrt{M_\beta} f^\circ \geq 0, \qquad f^\circ \in H^s(\mathbb{R}^d),$$

satisfying smallness and centering conditions,

$$\|f^\circ\|_{H^s(\mathbb{R}^d)} \leq \frac{1}{C_{V,\beta,s}}, \qquad \int_{\mathbb{R}^d} \left(1, v, \tfrac{1}{2}|v|^2\right) \sqrt{M_\beta} f^\circ = 0,$$

there exists unique global strong solution F with initial data F°

$$F = M_\beta + \sqrt{M_\beta} f \geq 0, \qquad f \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^d)),$$

and it satisfies for all $t \geq 0$,

$$\|f^t\|_{H^s(\mathbb{R}^d)} \lesssim_{V,\beta,s} \|f^\circ\|_{H^s(\mathbb{R}^d)}.$$

Microscopic dynamics:
$N \gg 1$ particles $(X_i, V_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\phi(x) = |x|^{-1}$ Coulomb potential

$$\dot{X}_i(t) = V_i(t), \quad \dot{V}_i(t) = -\sum_{i \neq j} \frac{\theta_i \theta_j}{m_i} \nabla \phi(X_i - X_j).$$

Effective equation depends on:

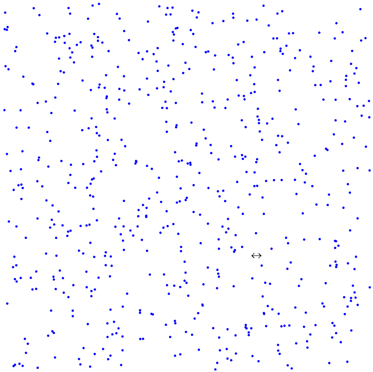
- temperature
- density of particles
- charges θ_i, masses m_i
- initial data/external fields

Well-posedness issue in attractive case, even $N = 3$

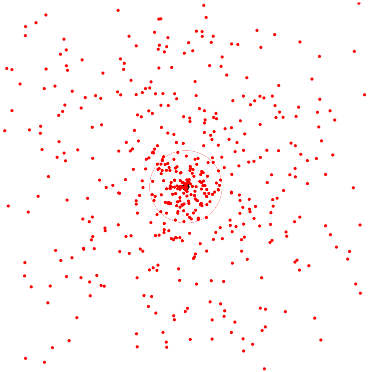
Observation: Coulomb pot. $\phi(x) = |x|^{-1}$ slow, non-integrable decay.

Question: Effect of single charge has infinite range?

Experiment: Measure effect of single ion on plasma



Particle system before perturbation



After perturbation through Ion at the center

Effective interaction decays exponentially over the Debye length

$$\lambda_D = \sqrt{\frac{T}{N\theta^2}}$$

Requires electroneutrality

- Either two species with positive/negative charge, or
- homogeneous background charge

First mathematically rigorous result (in some regimes) [Brydges, Federbush, 1980]
$\rightsquigarrow$ potential connection to [Kesten, Papanicolaou 1980]?.

- state-of-the art Coulomb gas theory, see [Serfaty, 2023]
- recent results extend to Riesz-gas, see Boursier, Leblé, Serfaty

Formal argument

Assuming screening (hom. background) + formal closure of BBGKY
One-particle density can be approximated by

$$\partial_t f(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} B(v, v - v')(\nabla f(v) f(v') - \nabla f(v') f(v)) dv' \right)$$

$$B(v, v - v'; \nabla f) = \int_{|k| \leq r_0} \frac{(k \otimes k)|\hat{\phi}(k)|^2}{|\epsilon(k, k \cdot v)|^2} \delta\big(k \cdot (v - v')\big) dk$$

where $r_0 = \frac{\theta^2}{mT}$. Integral divergent for large $|k| \gg 1$, more precisely

$$B \approx \log(N_D) B_{\mathrm{Landau}},$$

where N_D is number of particles in Debye sphere.

Theorem [Arroyo-Rabasa, W., 2021/2023]

Let $\mu \in \mathcal{M}(\mathbb{R}^3)$ be compactly supported measure with finite total variation and

$$\theta = \int_{\mathbb{R}^3} \mu(dx).$$

Assume f_0 radial and Penrose stable. Then there exists weak solution f to

$$v \cdot \nabla_x f - \nabla_x Q \cdot \nabla_v f = 0, \quad \lim_{|x| \to \infty} f(x,v) = f_0(v)$$
$$-\Delta_x Q(x) = (\rho[f] - 1) + \mu \,,$$

satisfying the screening estimates

$$0 \le Q(x) \le \frac{C\theta e^{-\frac{|x|}{\lambda_D}}}{|x|}, \quad |1 - \rho[f]| \le C\theta e^{-\frac{|x|}{\lambda_D}} \,,$$

Further, if $\theta < 0$, there are infinitely many such solutions.

For conjectures and results on force fields, see [Nota, Simonella, Velázquez, 2021], [Nota, Velázquez, W., 2022,2023].

- Understand equilibrium ensembles for ϕ power law
- Derivation of linear Boltzmann/Landau for power law potentials
- Derivation of linear Balescu-Lenard on kinetic scale
- Derivation of (linear) Boltzmann in (self-consistent) field
- PDE theory for (Coulomb) Balescu-Lenard

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Thank you!