At the transition between pulled and pushed fronts

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Inhomogeneous Random Systems

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Cross-Overs of Bramson's Shift at the Transition Between Pulled and Pushed Fronts

J. An, C. Henderson, L Ryzhik preprints 2021-2022

Pushed, pulled and pushmi-pullyu fronts of the Burgers-FKPP equation

Quantitative steepness, semi-FKPP reactions, and pushmi-pullyu fronts

Outline

Mean-field equations for

- the N-BBM (BBM= Branching Brownian Motion)
- the Fleming-Viot process
- Reaction-diffusion traveling equations

Pulled versus pushed fronts

The transition between pulled and pushed fronts

The large N corrections

Mean-field equations for

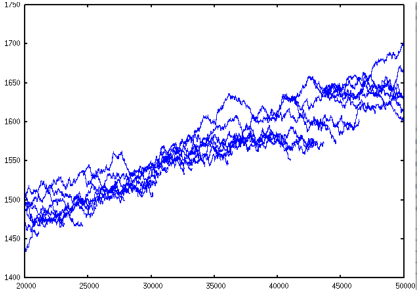
the N-BBM

the Fleming-Viot process

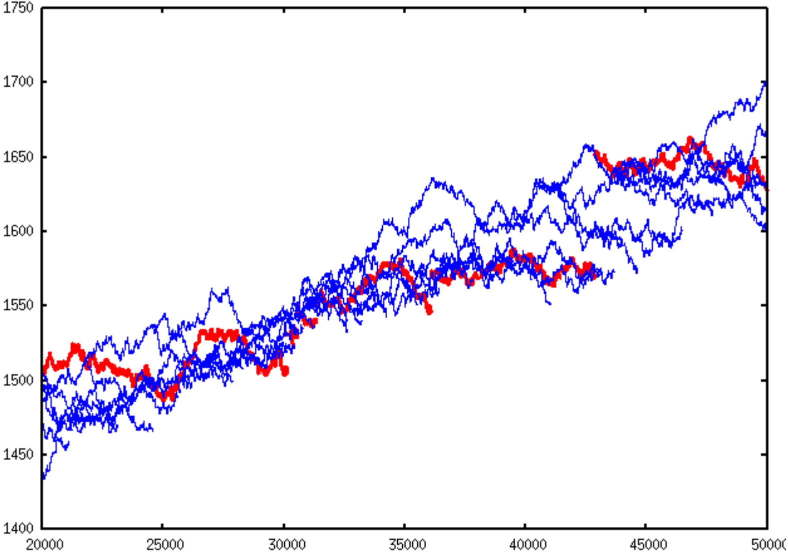
Reaction-diffusion traveling equations

The N-BBM

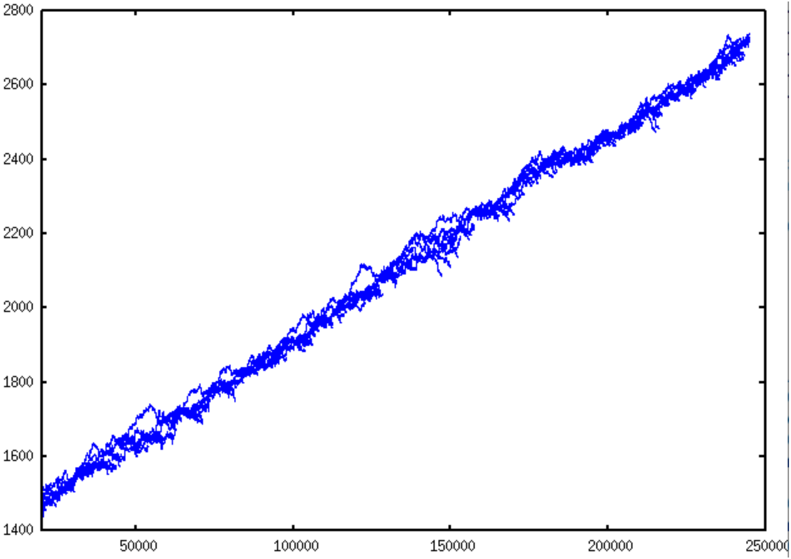
- N particles on a 1-d lattice
- The particles perform independent random walks
- The particles branch at rate λ
- At each branching event, the particle with lowest position is removed



Positions of the particles versus time



Positions of the particles versus time



Positions of the particles versus time

n_i = the number of particles at the right of site i

$$\frac{d\langle n_i\rangle}{dt} = \left\langle \left(n_{i-1} + (\lambda - 2)n_i + n_{i+1}\right) \Theta(N - n_i)\right\rangle$$

$\Theta(n) = 1$ if $n \geq 1$ and 0 otherwise (non-linear evolution)

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Mean-field approximation : $u_i = \frac{\langle n_i\rangle}{N}$

$$\begin{aligned} \frac{du_i}{dt} &= u_{i-1} + (\lambda - 2)u_i + u_{i+1} && \text{if} \quad u_i < 1 \\ &= 0 && \text{if} \quad u_i = 1 \end{aligned}$$

(still non-linear)

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(still non-linear)

Continuous space version of the mean field approximation:

$$\frac{du}{dt} = \begin{cases} \frac{d^2 u}{dx^2} + u & \text{if} \quad u < 1 \\ 0 & \text{if} \quad u = 1 \end{cases}$$

Questions: the N-BBM

Continuous space version (and the mean field approximation)

describes the large N limit

a continuous family of steady state solutions

the solution selected depends on the initial condition

logarithmic Bramson's shift of the front position

Large N corrections

one velocity selected

$$v_\infty - v_N \simeq \frac{A}{\log^2 N} \quad ; \quad D_N \simeq \frac{B}{\log^3 N}$$

the N-BBM

$$\frac{du}{dt} = \begin{cases} \frac{d^2u}{dx^2} + u & \text{if} \quad u < 1 \\ 0 & \text{if} \quad u = 1 \end{cases}$$

One family of traveling wave solutions

indexed by the velocity v

$$u = W_v(x - vt) \qquad \text{for} \qquad v \geq 2$$

$$W_v(x) = \frac{\gamma_2 e^{-\gamma_1 x} - \gamma_1 e^{-\gamma_2 x}}{\gamma_2 - \gamma_1}$$

where γ_1 and γ_2 are the two roots of $\gamma^2 + v\gamma + 1 = 0$

the N-BBM

$$\frac{du}{dt} = \begin{cases} \frac{d^2u}{dx^2} + u & \text{if} \quad u < 1 \\ 0 & \text{if} \quad u = 1 \end{cases}$$

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Bramson's shift

For an initial condition decaying fast enough $(\int u(x,0)\, x\, e^x\, dx < \infty)$

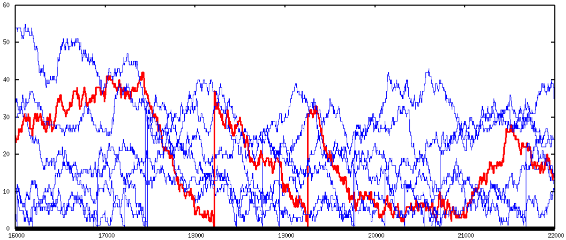
$$u(x + X_t, t) \to W_2(x) \qquad \text{as} \qquad t \to \infty$$

with

$$X_t = 2t - \frac{3}{2}\log t + A(\{u(x,0)\}) + O(t^{-1/2})$$

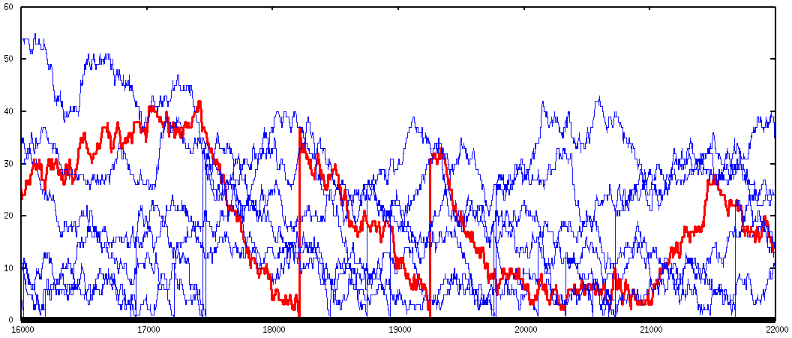
The Fleming-Viot process

- N particles on a 1-d lattice (on the positive side)
- The particles perform independent random walks biaised towards the origin
- When a particle hits the origin, it jumps to the position of one of the $N-1$ remaining particles



Positions of the particles versus time

The Fleming-Viot process



Positions of the particles versus time

The Fleming-Viot process

n_i = the number of particles at the right of site i

$$\alpha < \beta \quad ; \quad n_0 = 0$$

$$\frac{d\langle n_i \rangle}{dt} = \langle \alpha n_{i-1} - (\alpha + \beta) n_i + \beta n_{i+1} \rangle + \frac{\beta}{N} \langle n_1 n_i \rangle$$

The Fleming-Viot process

n_i = the number of particles at the right of site i

$$\alpha < \beta \quad ; \quad n_0 = 0$$

$$\frac{d\langle n_i \rangle}{dt} = \langle \alpha n_{i-1} - (\alpha + \beta) n_i + \beta n_{i+1} \rangle + \frac{\beta}{N} \langle n_1 n_i \rangle$$

mean-field approximation : $\quad u_i = \frac{\langle n_i \rangle}{N} \quad ; \quad \sum_i u_i = 1$

$$\frac{du_i}{dt} = \alpha\, u_{i-1} - (\alpha + \beta)\, u_i + \beta\, u_{i-1} + \alpha\, u_i\, u_1$$

A one family of steady state solutions indexed by u_1.

The Fleming-Viot process

mean field equation

Q_t is the number of hits of the origin during time t

If the initial condition $u_i(0)$ decays fast enough, one steady state solution is selected

$$Q_t = (\sqrt{\beta} - \sqrt{\alpha})^2 \ t + \frac{3}{2} \log t + A(\{u_k(0)\}) + o(1)$$

with

$$e^{-A} = \frac{1}{\sqrt{\pi}\,(\beta\alpha)^{1/4}\,(\sqrt{\beta} - \sqrt{\alpha})^2} \sum_{k \geq 1} k \left(\frac{\beta}{\alpha}\right)^{k/2} u_k(0)$$

for the FV process, Q_t is explicit at all times

The Fleming-Viot process

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for the FV process, Q_t is explicit at all times

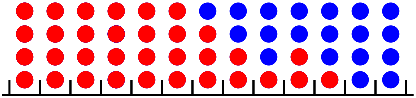
Large N corrections

one flux selected

$$\frac{Q_t}{t} - (\sqrt{\beta} - \sqrt{\alpha})^2 \simeq \frac{A}{\log^2 N} \quad \text{numerics} \quad ; \quad \frac{\langle Q_t^2 \rangle - \langle Q_t \rangle^2}{t} \simeq \frac{B}{\log^3 N} \quad \text{guess}$$

Reaction-diffusion traveling equations

- N particles on each site
- On site x, $n(x,t)$ red particles and $N - n(x,t)$ blue particles
- exchange of particles between neighboring sites
- reation : $n(x) \to n(x) + 1$ with robability f(n(x)/N)



$$u(x,t) = \frac{n(x,t)}{N}$$

meanfield equation + continuous limit

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

For $f(u) = u(1-u)$: Fisher-KPP equation

Fisher-KPP fronts

$f(0) = f(1) = 0$; $f'(0) = 1$; $f(u) > 0$ for $0 < u < 1$; and $f(u) < u$

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

$u = 0$ unstable ; $u = 1$ stable

One parameter family of travelling wave solutions

The solution is selected by the initial conditions

For initial conditions decaying fast enough

$$X_t = 2\ t - \frac{3}{2}\ \log t + A(\{u(x,0)\}) + o(1)$$

Pulled versus pushed fronts

Pulled versus pushed fronts

$f(0) = f(1) = 0$; f'(0)=1 $f(u) > 0$ for $0 < u < 1$; and ~~$f(u) < u$~~

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

For example $f(u) = (u - u^2)(1 + 2au)$ Hadeler, Rothe 1975

Pulled versus pushed fronts

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$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

For example $f(u) = (u - u^2)(1 + 2au)$ Hadeler, Rothe 1975

- pulled front $a < 1$

$$X_t = 2t - \frac{3}{2}\log t + O(1)$$

- the transition point $a = 1$

$$X_t = 2t - \frac{1}{2}\log t + O(1)$$

- pushed front $a > 1$

$$X_t = \left(a^{\frac{1}{2}} + a^{-\frac{1}{2}}\right) t + O(1)$$

Pulled versus pushed fronts

$f(0) = f(1) = 0$; f'(0)=1 $f(u) > 0$ for $0 < u < 1$; and ~~$f(u) < u$~~

$$\frac{du}{dt} = \frac{d^2 u}{dx^2} + f(u)$$

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- the transition point $a = 1$

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- pushed front $a > 1$

$$X_t = \left(a^{\frac{1}{2}} + a^{-\frac{1}{2}} \right) t + O(1)$$

Other examples $f(u) = (u - B(u))(1 + aB'(u))$ An, Henderson, Ryzhik, 2022

with $B(0) = B'(0) = 0$ and $B(1) = 1$

The vanishing of an amplitude

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

A traveling wave solution $u(x,t) = W_v(x - vt)$
W is solution of

$$W_v'' + vW_v' + f(W_v) = 0$$

The vanishing of an amplitude

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + f(u)$$

A traveling wave solution $u(x,t) = W_v(x - vt)$

W is solution of

$$W_v'' + vW_v' + f(W_v) = 0$$

For large x

$$W_v(x) \sim A_1(v)e^{-\gamma_1 x} + A_2(v)e^{-\gamma_2 x} \quad \text{with} \quad \gamma_1 \leq \gamma_2$$

and γ_1 and γ_2 are solutions of

$$\gamma^2 - v\gamma + f'(0) = 0$$

The vanishing of an amplitude

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If $A_1(v)$ vanishes at some $v^* > v_{\min}$, the front is pushed and

$$X_t = v^* t + O(1)$$

An exactly solvable case : the modified N-BBM

$$u(X_t, t) = 1 \qquad \text{for} \quad x \leq X_t$$

$$\frac{du}{dt} = \frac{d^2 u}{dx^2} + u \qquad \text{for} \quad x > X_t$$

$$\partial_x u(X_t, t) = -a \qquad \text{for the N-BBM } a = 0$$

An exactly solvable case : the modified N-BBM

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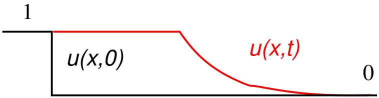
for the N-BBM $a = 0$

The traveling wave



$$W_v(x) = \frac{1 - a\,\gamma_1}{1 - \gamma_1^2} e^{-\gamma_1 x} + \frac{1 - a\,\gamma_2}{1 - \gamma_2^2} e^{-\gamma_2 x}$$

where $\gamma_1 < 1 < \gamma_2$ are the two roots of $\gamma^2 - v\gamma + 1 = 0$

An exactly solvable case : the modified N-BBM

$$
\begin{aligned}
u(X_t, t) &= 1 && \text{for} \quad x \le X_t \\
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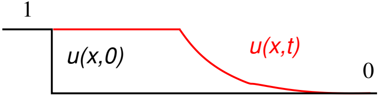
\end{aligned}
$$

for the N-BBM $a = 0$

The traveling wave



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where $\gamma_1 < 1 < \gamma_2$ are the two roots of $\gamma^2 - v\gamma + 1 = 0$

$$
X_t = \begin{cases}
2t - \frac{3}{2} \log t + O(1) & \text{for} \quad a < 1 \\
2t - \frac{1}{2} \log t + O(1) & \text{for} \quad a = 1 \\
X_t = \left(a + a^{-1}\right) t + O(1) & \text{for} \quad a > 1
\end{cases}
$$

The transition between pulled and pushed fronts

Cross-over $a \to 1$ and $t \to \infty$ for the modified N-BBM

For $a = 1 + \epsilon$ and large t

Initial condition $u(x, 0)$ decaying fast enough

$$X_t = 2t - \frac{1}{2} \log t + \Psi_1(\epsilon\sqrt{t}) + A(\{u(x,0)\}) + o(1)$$

where

$$\Psi_1(z) = \log \left[1 + 2ze^{z^2} \int_{-\infty}^{z} dv \ e^{-v^2} \right]$$

and

Cross-over $a \to 1$ and $t \to \infty$ for the modified N-BBM

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and

$$A\big(\{u(x,0)\}\big) = \log\left[1 + \int_0^{\infty} dz\ e^z\ u(z,0)\right] - \frac{1}{2}\log\pi$$

Cross-over $a \to 1$ and $t \to \infty$ for the modified N-BBM

For $a = 1 + \epsilon$ and large t

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$$A(\{u(x,0)\}) = \log \left[1 + \int_0^\infty dz \ e^z \ u(z,0) \right] - \frac{1}{2} \log \pi$$

Question : is this cross-over universal ?

Beresticky, Brunet, Derrida, 2017

$$1 = (1 - ar) \int_0^\infty dt\, e^{rX_t - (1+r^2)t}$$

Example: step initial condition $u_0(x) = 0$, and $a = 1 + \epsilon$.

$$1 = \text{r.h.s.}$$

At each order of $a - 1$ look for the most singular term as $r \to 1$

1. $X_t = 2t$

$$\text{r.h.s.} = \frac{1}{(1-r)} + \frac{\epsilon}{(1-r)^2} + \frac{\epsilon^2}{(1-r)^3} + \cdots$$

Beresticky, Brunet, Derrida, 2017

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2. $X_t = 2t - \frac{1}{2}\log t - \frac{1}{2}\log \pi$

$$\text{r.h.s.} = 1 + \frac{\epsilon}{(1-r)} + O(\epsilon^2)$$

Beresticky, Brunet, Derrida, 2017

$$1 = (1 - ar) \int_0^\infty dt\, e^{rX_t - (1+r^2)t}$$

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$$\text{r.h.s.} = \frac{1}{(1-r)} + \frac{\epsilon}{(1-r)^2} + \frac{\epsilon^2}{(1-r)^3} + \cdots$$

2. $X_t = 2t - \frac{1}{2}\log t - \frac{1}{2}\log \pi$

$$\text{r.h.s.} = 1 + \frac{\epsilon}{(1-r)} + O(\epsilon^2)$$

3. $X_t = 2t - \frac{1}{2}\log t - \frac{1}{2}\log \pi + \sqrt{\pi}\epsilon\sqrt{t} + \cdots$

$$\text{r.h.s.} = 1 + O(\epsilon^2) + \cdots$$

The cross-over for the Fleming-Viot process

n_i = the number of particles at the right of site i

$$\alpha < \beta \quad ; \quad n_0 = 0$$

$$\frac{d\langle n_i\rangle}{dt} = \langle \alpha n_{i-1} - (\alpha + \beta_i) n_i + \beta_{i-1} n_{i+1}\rangle + \frac{\beta_1}{N}\langle n_1 n_i\rangle$$

$\beta_i = \beta$ for $i \geq 2$
$\beta_1 \neq \beta$

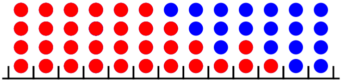
Cross-over

$$\beta_1 = \beta - \sqrt{\alpha\beta} - (\alpha\beta)^{\frac{1}{4}}\ \epsilon$$

$$Q_t = (\sqrt{\beta} - \sqrt{\alpha})^2 + \frac{1}{2}\log t - \Psi_1(\epsilon\sqrt{t}) + A(\{u(x,0)\}) + o(1)$$

The large N corrections

Reaction-diffusion problem



$$n(x,t) + n(x,t) = N$$

1. $N = \infty$: Fisher-KPP equation

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + u - u^2$$

2. N large : Noisy Fisher-KPP equation

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + u - u^2 + \sqrt{\frac{u(1-u)}{N}}\,\eta(x,t)$$

with $\eta(x,t)$ white noise

Traveling wave equation + noise

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + u - u^2 + \frac{1}{\sqrt{N}} \quad \eta(x,t)\sqrt{u(1-u)}$$

Brunet Derrida 1997

$$\frac{X_t}{t} \to v_N$$

Brunet Derrida Mueller Munier 2006

$$\frac{\langle X_t^2 \rangle - \langle X_t \rangle^2}{t} \to D_N$$

Mueller Mytnik Quastel 2008

$$v_N \simeq 2 - \frac{\pi^2}{\log^2 N} \qquad D_N \simeq \frac{2\pi^4}{3\log^3 N}$$

Noisy Fisher KPP equation

$$\frac{du}{dt} = \frac{d^2u}{dx^2} + u - u^2 + \sqrt{\epsilon\; u(1-u)}\;\eta(x,t)$$

- Reaction diffusion model: $A \to 2A$ and $2A \to A$ at rate ϵ

  $u(x,t)$ is the density

- N- Branching Brownian motion (selection: $\epsilon = N^{-1}$)

Doering, Mueller, Smereka 2003
Brunet Derrida Mueller Munier 2006-2007

For the N-BBM

For a population of fixed size N

- Large N limit (first $t \to \infty$, then $N \to \infty$) becomes the free boundary problem

Durrett, Remenik 2011
De Masi, Ferrari, Presutti, Soprano-Loto 2019

$$\begin{aligned} u(X_t, t) &= 1 && \text{for} \quad x \le X_t \\ \frac{du}{dt} &= \frac{d^2 u}{dx^2} + u && \text{for} \quad x > X_t \end{aligned}$$

- Velocity as a function of N

$$v_N \simeq 2 - \frac{\pi^2}{\log^2 N} + \cdots$$

Bérard Gouéré 2010

Crossover between pulled and pushed fronts

- $a < 1$ (pulled case)

Brunet Derrida 1997

$$v_N = 2 - \frac{\pi^2}{\log^2 N} + \cdots$$

Crossover between pulled and pushed fronts

- $a < 1$ (pulled case) Brunet Derrida 1997

$$v_N = 2 - \frac{\pi^2}{\log^2 N} + \cdots$$

- $a > 1$ (pushed case) Kessler Ner Sander 1998

$$v_N = a + a^{-1} - \frac{(1-a^2)^2}{a^3} N^{a-a^{-1}} + \cdots$$

Crossover between pulled and pushed fronts

- $a < 1$ (pulled case) Brunet Derrida 1997

$$v_N = 2 - \frac{\pi^2}{\log^2 N} + \cdots$$

- $a > 1$ (pushed case) Kessler Ner Sander 1998

$$v_N = a + a^{-1} - \frac{(1-a^2)^2}{a^3} N^{a-a^{-1}} + \cdots$$

- $a = 1$ transition

$$v_N = 2 - \frac{\pi^2}{4 \log^2 N} + \cdots$$

Crossover between pulled and pushed fronts

- $a < 1$ (pulled case) Brunet Derrida 1997

$$v_N = 2 - \frac{\pi^2}{\log^2 N} + \cdots$$

- $a > 1$ (pushed case) Kessler Ner Sander 1998

$$v_N = a + a^{-1} - \frac{(1-a^2)^2}{a^3} N^{a-a^{-1}} + \cdots$$

- $a = 1$ transition

$$v_N = 2 - \frac{\pi^2}{4 \log^2 N} + \cdots$$

- cross-over $a \to 1$ and $\epsilon \to 0$

$$v_N = 2 - \frac{\chi^2}{\log^2 N} + \cdots$$

where χ is solution of $\chi = (\epsilon \ \log N) \tan(\chi)$

Conclusion

- Universality of these cross-overs
- In particular

$$v_N = \begin{cases} 2 - \frac{\pi^2}{\log^2 N} & \text{pulled front} \\ 2 - \frac{\pi^2}{4\log^2 N} & \text{transition} \end{cases}$$

- Numerical, theoretical and mathematical works needed
- Genealogies of Fleming Viot process
- Tip if the BBM ? $=$? tip of Fleming Viot process