Mean-field limits for log/Coulomb/Riesz interacting diffusions

Sylvia SERFATY

Courant Institute, NYU

IRS2024
January 23, 2024

The discrete coupled ODE system

Consider

$$H_N(x_1,\dots,x_N)=\frac{1}{2}\sum_{1\le i\ne j\le N}\mathrm{g}(x_i-x_j),\qquad x_i\in\mathbb{R}^d$$

Model case:

$$\begin{cases}\mathrm{g}(x)=\dfrac{1}{s|x|^s} & d-2\le s<d \quad \textbf{Riesz case}\\ \mathrm{g}(x)=-\log|x| & s=0 \quad \textbf{log case}\end{cases}$$

The discrete coupled ODE system

Consider

$$H_N(x_1, \dots, x_N) = \frac{1}{2} \sum_{1 \le i \ne j \le N} g(x_i - x_j), \qquad x_i \in \mathbb{R}^d$$

Model case:

$$\begin{cases} g(x) = \dfrac{1}{s|x|^s} & d-2 \le s < d \quad \textbf{Riesz case} \\ g(x) = -\log|x| & s = 0 \quad \textbf{log case} \end{cases}$$

Evolution equation

$$\dot{x}_i = -\frac{1}{N} \left(\nabla_i H_N(x_1, \dots, x_N) \right) + \frac{1}{\sqrt{\beta}} dW_i^t \qquad \textbf{gradient flow}$$

$$\dot{x}_i = -\frac{1}{N} \mathbb{J} \nabla_i H_N(x_1, \dots, x_N) + \frac{1}{\sqrt{\beta}} dW_i^t \qquad \textbf{conservative flow} \quad (\mathbb{J}^T = -\mathbb{J})$$

$$\ddot{x}_i = -\frac{1}{N} \nabla_i H_N(x_1, \dots, x_N) + \frac{1}{\sqrt{\beta}} dW_i^t \qquad \textbf{Newton's law}$$

possibly with added noise $1/\sqrt{\beta} dW_i^t$, N independent Brownian motions, β= inverse temperature

Questions

For a general system

$$\dot{x}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) + \frac{1}{\sqrt{\beta}} dW_i^t$$

- What is the limit of the **empirical measure**? Is there μ^t such that for each t

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^t} \rightharpoonup \mu^t \tag{1}$$

Questions

For a general system

$$\dot{x}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) + \frac{1}{\sqrt{\beta}} dW_i^t$$

- What is the limit of the **empirical measure**? Is there μ^t such that for each t

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^t} \rightharpoonup \mu^t \tag{1}$$

- if $f_N^0(x_1, \ldots, x_N)$ is the probability density of position of the system at time 0, what is the limit behavior of f_N^t?

Questions

For a general system

$$\dot{x}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) + \frac{1}{\sqrt{\beta}} dW_i^t$$

- What is the limit of the **empirical measure**? Is there μ^t such that for each t

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^t} \rightharpoonup \mu^t \qquad (1)$$

- if $f_N^0(x_1, \dots, x_N)$ is the probability density of position of the system at time 0, what is the limit behavior of f_N^t?
- **propagation of chaos** (Boltzmann, Kac, Dobrushin): if $f_N^0(x_1, \dots, x_N) \simeq \mu^0(x_1) \dots \mu^0(x_N)$ is it true that

$$f_N^t(x_1, \dots, x_N) \simeq \mu^t(x_1) \dots \mu^t(x_N)?$$

in the sense of convergence of the k-point marginal $f_{N,k}$.

Formal limit

Use

$$\partial_t \delta_{x(t)} + \mathrm{div}\,\left(\dot{x}\delta_{x(t)}\right) = 0$$

or Liouville equation + BBGKY hierarchy

$$\partial_t f_N + \sum_{i=1}^N \nabla_{x_i}\left(f_N \frac{1}{N}\sum_{i=1}^N K(x_i - x_j)\right) = 0$$

We *formally* expect $\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \rightharpoonup \mu^t$ where μ^t solves the mean-field equation

$$\boxed{\partial_t \mu = \mathrm{div}\,((K * \mu)\mu) + \frac{1}{2\beta}\Delta\mu} \quad \text{(MF)}$$

or in the second order case the Vlasov / McKean-Vlasov equation

$$\boxed{\partial_t \rho + v \cdot \nabla_x \rho + (K * \mu) \cdot \nabla_v \rho + \frac{1}{2\beta}\Delta\rho = 0 \qquad \mu = \int_{\mathbb{R}^d} \rho(x, v)dv}$$

How to prove convergence?

- **Trajectorial** method ([Sznitman]...): compare true trajectories to those following characteristics for the limit equation. OK for K Lipschitz.

How to prove convergence?

- **Trajectorial** method ([Sznitman]...): compare true trajectories to those following characteristics for the limit equation. OK for K Lipschitz.
- Find a good metric, typically Wasserstein W_1 such that

$$\partial_t W_1(\mu_1(t), \mu_2(t)) \leq C W_1(\mu_1(t), \mu_2(t))$$

  for two solutions of the mean-field evolution. Apply to μ_N^t and μ^t. [Braun-Hepp, Dobrushin, Neunzert-Wick...]

How to prove convergence?

- **Trajectorial** method ([Sznitman]...): compare true trajectories to those following characteristics for the limit equation. OK for K Lipschitz.
- Find a good metric, typically Wasserstein W_1 such that

$$\partial_t W_1(\mu_1(t), \mu_2(t)) \leq C W_1(\mu_1(t), \mu_2(t))$$

  for two solutions of the mean-field evolution. Apply to μ_N^t and μ^t. [Braun-Hepp, Dobrushin, Neunzert-Wick...]
- Use a **relative entropy method**: show a Gronwall relation for

$$0 \leq \mathcal{H}_N(f_N | \rho^{\otimes N}) := \frac{1}{N} \int f_N \log \frac{f_N}{\rho^{\otimes N}} dx_1 \dots dx_N.$$

  [Jabin-Wang '16] for $\beta < \infty$, K not too irregular.

How to prove convergence?

- **Trajectorial** method ([Sznitman]...): compare true trajectories to those following characteristics for the limit equation. OK for K Lipschitz.
- Find a good metric, typically Wasserstein W_1 such that

$$\partial_t W_1(\mu_1(t), \mu_2(t)) \leq C W_1(\mu_1(t), \mu_2(t))$$

  for two solutions of the mean-field evolution. Apply to μ_N^t and μ^t. [Braun-Hepp, Dobrushin, Neunzert-Wick...]
- Use a **relative entropy method**: show a Gronwall relation for

$$0 \leq \mathcal{H}_N(f_N|\rho^{\otimes N}) := \frac{1}{N}\int f_N \log \frac{f_N}{\rho^{\otimes N}} dx_1 \dots dx_N.$$

  [Jabin-Wang '16] for $\beta < \infty$, K not too irregular.
- In the Coulomb case, results in 2d for first order evolutions [Goodman-Hou-Lowengrub '90, Schochet '96,Osada '87, Fournier-Hauray-Mischler '14, Duerinckx '15] but for $d \geq 3$ open.

How to prove convergence?

- **Trajectorial** method ([Sznitman]...): compare true trajectories to those following characteristics for the limit equation. OK for K Lipschitz.
- Find a good metric, typically Wasserstein W_1 such that
$$\partial_t W_1(\mu_1(t), \mu_2(t)) \leq C W_1(\mu_1(t), \mu_2(t))$$
for two solutions of the mean-field evolution. Apply to μ_N^t and μ^t. [Braun-Hepp, Dobrushin, Neunzert-Wick...]
- Use a **relative entropy method**: show a Gronwall relation for
$$0 \leq \mathcal{H}_N(f_N|\rho^{\otimes N}) := \frac{1}{N} \int f_N \log \frac{f_N}{\rho^{\otimes N}} dx_1 \dots dx_N.$$
[Jabin-Wang '16] for $\beta < \infty$, K not too irregular.
- In the Coulomb case, results in 2d for first order evolutions [Goodman-Hou-Lowengrub '90, Schochet '96,Osada '87, Fournier-Hauray-Mischler '14, Duerinckx '15] but for $d \geq 3$ open.
- for convergence to Vlasov-Poisson/Riesz
  - [Hauray-Jabin '15, Jabin-Wang '17] $s < d - 2$, Coulomb interaction (or more singular) remains open.
  - [Boers-Pickl '16, Lazarovici '16, Lazarovici-Pickl '17] Coulomb with N-dependent cut-off

The modulated energy method

Idea: use **Riesz-based metric**:

$$\|\mu - \nu\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{g}(x-y) d(\mu-\nu)(x) d(\mu-\nu)(y).$$

Observe weak-strong uniqueness property of the solutions to (MF) for $\|\cdot\|$:

$$\|\mu_1^t - \mu_2^t\|^2 \leq e^{Ct} \|\mu_1^0 - \mu_2^0\|^2 \qquad C = C(\|\nabla^2(\mathrm{g} * \mu_2)\|_{L^\infty})$$

In the discrete case, let X_N denote $(x_1, \dots, x_N)$ and take for modulated energy,

$$F_N(X_N, \mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \mathrm{g}(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)(x) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)(y)$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, and μ^t solves (MF).

Theorem (S '18 ($d-2 \le s < d$), H.Q.Nguyen-Rosenzweig-S '21 $s < d$)

Case $\beta = \infty$. Assume (MF) admits a solution $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$ with

$$\begin{cases} \nabla^2 g * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)) \text{ in the Coulomb case} \\ \text{additional regularity conditions on } \mu^t \text{ in the other cases} \end{cases}$$

Theorem (S '18 ($d-2 \leq s < d$), H.Q.Nguyen-Rosenzweig-S '21 $s < d$)

Case $\beta = \infty$. Assume (MF) admits a solution $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$ with

$$\begin{cases} \nabla^2 \mathrm{g} * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)) \text{ in the Coulomb case} \\ \text{additional regularity conditions on } \mu^t \text{ in the other cases} \end{cases}$$

There exist constants C_1, C_2 depending on the norms of μ^t and $\gamma > 0$ depending on d, s, σ, s.t. $\forall t \in [0,T]$

$$|F_N(X_N^t, \mu^t)| \leq \left(|F_N(X_N^0, \mu^0)| + C_1 N^{-\gamma}\right) e^{C_2 t}.$$

Theorem (S '18 ($d-2 \leq s < d$), H.Q.Nguyen-Rosenzweig-S '21 $s < d$)

Case $\beta = \infty$. Assume (MF) admits a solution $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$ with

$$\begin{cases} \nabla^2 \mathrm{g} * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)) \text{ in the Coulomb case} \\ \text{additional regularity conditions on } \mu^t \text{ in the other cases} \end{cases}$$

There exist constants C_1, C_2 depending on the norms of μ^t and $\gamma > 0$ depending on d, s, σ, s.t. $\forall t \in [0,T]$

$$|F_N(X_N^t, \mu^t)| \leq \left(|F_N(X_N^0, \mu^0)| + C_1 N^{-\gamma}\right) e^{C_2 t}.$$

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ and is such that $\lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0$, then the same is true for every $t \in [0,T]$ and $\mu_N^t \rightharpoonup \mu^t$

Theorem (S '18 ($d-2 \le s < d$), H.Q.Nguyen-Rosenzweig-S '21 $s < d$)

Case $\beta = \infty$. Assume (MF) admits a solution $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$ with

$$\begin{cases} \nabla^2 \mathrm{g} * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)) \text{ in the Coulomb case} \\ \text{additional regularity conditions on } \mu^t \text{ in the other cases} \end{cases}$$

There exist constants C_1, C_2 depending on the norms of μ^t and $\gamma > 0$ depending on d, s, σ, s.t. $\forall t \in [0,T]$

$$|F_N(X_N^t, \mu^t)| \le \left(|F_N(X_N^0, \mu^0)| + C_1 N^{-\gamma}\right) e^{C_2 t}.$$

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ and is such that $\lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0$, then the same is true for every $t \in [0,T]$ and $\mu_N^t \rightharpoonup \mu^t$

- [Rosenzweig '20] improved the result in the Coulomb case: $\mu^0 \in L^\infty$ suffices (for short times if $d \ge 3$).
- Now sharp rate $N^{-\gamma} = N^{\frac{s}{d}-1}$ [Rosenzweig-S '23]

Convergence to Vlasov-Poisson in the monokinetic case

Let $Z_N = ((x_1, v_1), \ldots, (x_N, v_N))$ where $v_i = \dot{x}_i$.
Monokinetic version of (VP) (pressureless Euler-Poisson):

$$\rho^t(x, v) = \mu^t(x)\delta_{v=u^t(x)}$$

$$\boxed{\partial_t \mu + \mathrm{div}\,(\mu u) = 0 \qquad \partial_t u + u \cdot \nabla u = -\nabla \mathrm{g} * \mu} \qquad (PEP)$$

Convergence to Vlasov-Poisson in the monokinetic case

Let $Z_N = ((x_1, v_1), \ldots, (x_N, v_N))$ where $v_i = \dot{x}_i$.
Monokinetic version of (VP) (pressureless Euler-Poisson):

$$\rho^t(x, v) = \mu^t(x)\delta_{v=u^t(x)}$$

$$\boxed{\partial_t \mu + \mathrm{div}\,(\mu u) = 0 \qquad \partial_t u + u \cdot \nabla u = -\nabla \mathrm{g} * \mu} \qquad (PEP)$$

Use modulated energy

$$E_N(Z_N, (\mu, u)) := \frac{1}{N} \sum_{i=1}^{N} |u(x_i) - v_i|^2 + F_N(X_N, \mu)$$

Convergence to Vlasov-Poisson in the monokinetic case

Let $Z_N = ((x_1, v_1), \ldots, (x_N, v_N))$ where $v_i = \dot{x}_i$.
Monokinetic version of (VP) (pressureless Euler-Poisson):

$$\rho^t(x,v) = \mu^t(x)\delta_{v=u^t(x)}$$

$$\boxed{\partial_t \mu + \mathrm{div}\,(\mu u) = 0 \qquad \partial_t u + u \cdot \nabla u = -\nabla \mathrm{g} * \mu} \qquad (PEP)$$

Use modulated energy

$$E_N(Z_N, (\mu, u)) := \frac{1}{N}\sum_{i=1}^N |u(x_i) - v_i|^2 + F_N(X_N, \mu)$$

Theorem (Duerinckx-S '18)

Assume Z_N^t solves Newton's law with initial data Z_N^0. Assume (μ, u) is a sufficiently regular solution to (PEP) on $[0, T]$. Then

$$E_N(Z_N^t, (\mu^t, u^t)) \le \left(E_N(Z_N^0, (\mu^0, u^0)) + N^{-\beta}\right) e^{C_2 t}$$

In particular if $\lim_{N\to\infty} E_N(Z_N^0, (\mu^0, u^0)) = 0$ then $\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \rightharpoonup \mu^t$ for all $t \in [0, T]$.

The commutator estimate / main functional inequality

When computing $\frac{d}{dt}F_N(X_N^t,\mu^t)$ we find we need to control

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle}(\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y)d(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu)^{\otimes 2}(x,y)$$

with $\psi=\nabla \mathrm{g}*\mu^t$ or $\mathbb{J}\nabla \mathrm{g}*\mu^t$.

The commutator estimate / main functional inequality

Proposition (S, NRS)

All Riesz-like cases $s < d$.

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (\psi(x) - \psi(y)) \cdot \nabla \mathrm{g}(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y)$$

$$\leq C\|D\psi\|_{L^\infty}(F_N(X_N,\mu) + N^{-\gamma}),$$

The commutator estimate / main functional inequality

Proposition (S, NRS)

All Riesz-like cases $s < d$.

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (\psi(x) - \psi(y)) \cdot \nabla \mathrm{g}(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)^{\otimes 2}(x,y) \le C\|D\psi\|_{L^\infty}(F_N(X_N, \mu) + N^{-\gamma}),$$

Why commutator ? Let $f = \frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu$

$$\int \psi \cdot \nabla(\mathrm{g} * f) - \mathrm{g} * (\nabla \cdot (\psi f)) = \langle f, \left[\psi, \frac{\nabla}{(-\Delta)^{\frac{d-s}{2}}}\right] f \rangle_{L^2}$$

Topic of *singular integrals / Christ-Journé operators*, [Hadzic, Seeger, Smart, Street] (all $\mathrm{div}\ \psi = 0$).

The commutator estimate / main functional inequality

Proposition (S, NRS)

All Riesz-like cases $s < d$.

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} (\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)^{\otimes 2}(x,y) \le C\|D\psi\|_{L^\infty}(F_N(X_N,\mu)+N^{-\gamma}),$$

Why commutator ? Let $f=\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu$

$$\int \psi\cdot\nabla(\mathrm{g}*f)-\mathrm{g}*(\nabla\cdot(\psi f)) = \langle f, \Big[\psi, \frac{\nabla}{(-\Delta)^{\frac{d-s}{2}}}\Big] f\rangle_{L^2}$$

Topic of *singular integrals / Christ-Journé operators*, [Hadzic, Seeger, Smart, Street] (all $\mathrm{div}\ \psi=0$).
Estimate used to treat the *quantum* Coulomb mean-field limit [Golse-Paul, Rosenzweig], *quasi-neutral* limits [Iacobelli-Han Kwan, Rosenzweig, RS, Ben Porat]

With noise : the modulated free energy

- If $\beta < \infty$, [Bresch-Jabin-Wang '19] incorporate the modulated energy into their relative entropy method: use a **modulated free energy**

$$\mathcal{F}_N^\beta(f_N, \mu) = \frac{1}{\beta}\mathcal{H}_N(f_N|\mu^{\otimes N}) + \int f_N(X_N)F_N(X_N, \mu)dx_1 \dots dx_N$$

  where f_N = probability density on configurations.
  Dissipative case only
- commutator estimate then suffices to complete the proof
- allows to treat Riesz cases $s < d$ and attractive logarithmic interactions $\rightsquigarrow$ convergence to Patlak-Keller-Segel

Evolution of modulated free energy

Introduce **modulated Gibbs measure**

$$\mathbb{Q}_{N,\beta}(\mu) = \frac{1}{K_{N,\beta}(\mu)} e^{-\beta N F_N(X_N,\mu)} d\mu(x_1)\dots d\mu(x_N)$$

Then [Rosenzweig-S '23]

$$\mathcal{F}_N^\beta(f_N,\mu) = \frac{1}{\beta}\mathcal{H}_N(f_N|\mathbb{Q}_{N,\beta}(\mu)) + \underbrace{\frac{1}{N}\log K_{N,\beta}(\mu)}_{\text{o(1) constant}}$$

Evolution of modulated free energy

Introduce **modulated Gibbs measure**

$$\mathbb{Q}_{N,\beta}(\mu) = \frac{1}{K_{N,\beta}(\mu)} e^{-\beta N F_N(X_N,\mu)} d\mu(x_1)\dots d\mu(x_N)$$

Then [Rosenzweig-S '23]

$$\mathcal{F}_N^\beta(f_N,\mu) = \frac{1}{\beta}\mathcal{H}_N(f_N|\mathbb{Q}_{N,\beta}(\mu)) + \underbrace{\frac{1}{N}\log K_{N,\beta}(\mu)}_{\text{o(1) constant}}$$

$$\frac{d}{dt}\mathcal{F}_N^\beta(f_N^t,\mu^t) = -\frac{1}{\beta^2 N}\underbrace{\int\left|\nabla\sqrt{\frac{f_N^t}{\mathbb{Q}_{N,\beta}(\mu^t)}}\right|^2 d\mathbb{Q}_{N,\beta}(\mu^t)}_{\text{relative Fisher information}}$$

$$-\frac{1}{2}\int df_N^t \underbrace{\iint_{\triangle^c}(v^t(x)-v^t(y))\cdot\nabla \mathrm{g}(x-y)d\left(\frac{1}{N}\sum_{i=1}^N\delta_{x_i}-\mu^t\right)^{\otimes 2}(x,y)dxdy}_{\text{commutator term}}$$

where $v^t = \nabla \mathrm{g} * \mu^t + \frac{1}{\beta}\nabla\log\mu^t$

Global in time convergence?

- Use Fisher information term and assume uniform “modulated" **Log-Sobolev inequality**

$$N \underbrace{\mathcal{H}_N(f_N^t | \mathbb{Q}_{N,\beta}(\mu^t))}_{\text{relative entropy}} \leq C_{LS} \underbrace{\int \left| \nabla \sqrt{\frac{f_N^t}{\mathbb{Q}_{N,\beta}(\mu^t)}} \right|^2 d\mathbb{Q}_{N,\beta}(\mu^t)}_{\text{relative Fisher}}$$

then this + commutator estimate gives

$$\frac{d}{dt}\mathcal{F}_N^\beta \leq -C\mathcal{F}_N^\beta + o(1)$$

so exponential convergence to tensorized state (**generation of chaos**) [Rosenzweig-S '23]

Global-in-time convergence?

- [Guillin, Le Bris, Monmarché '21]: Jabin-Wang's relative entropy approach combined with logarithmic Sobolev inequality to control the entropy dissipation (= Fisher information) by the relative entropy itself.
- [RS '21] take advantage of decay of solution / velocity field to obtain uniform in time convergence in the sub-Coulomb case.
- [Hess-Childs '23] obtains LDP around the mean-field limit in the same sub-Coulomb case
- prove and exploit the **decay** rate of ∇v^t as $t \to \infty$ to insert into optimized commutator estimate. Works in torus super-Coulomb setting with exponential decay rate [Chodron de Courcel-Rosenzweig-S '23]

Global in time result: subcoulomb case via modulated energy

Step 1. Prove that μ^t **decays** fast enough in long time thanks to the dissipation.

Proposition (RS)

Let $1 \leq p \leq q \leq \infty$.

$$\|\mu^t\|_{L^q} \leq C_{p,q} \left(\frac{2\pi t}{\beta(1/p - 1/q)} \right)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|\mu^0\|_{L^p}$$

Inspired by [Carlen-Loss] (case of divergence-free vector field).

Step 2. **Optimized** version of the functional inequality: use that $\psi^t = \mathbb{M}\nabla \mathrm{g} * \mu^t$

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (\psi^t(x) - \psi^t(y)) \cdot \nabla \mathrm{g}(x-y) d(\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i} - \mu^t)^{\otimes 2}(x,y)$$
$$\leq C \|\mu^t\|_{L^\infty}^{\frac{s+d}{2}} (F_N(X_N^t, \mu^t) + N^{-1+\frac{s}{d}}),$$

Theorem (Rosenzweig-S '21)

Assume $d \geq 3$, $0 < s < d-2$, *first order evolution with additive noise, dissipative or conservative. Assume* μ^t *is a global in time bounded solution to the limiting equation*

$$\partial_t \mu = -\mathrm{div}\,(\mu \mathbb{M} \nabla \mathrm{g} * \mu) + \frac{1}{2\beta}\Delta\mu.$$

Then

$$\mathbb{E}\left(|F_N(X_N^t, \mu^t)|\right) \leq C|F_N(X_N^0, \mu^0)| + N^{-\gamma}.$$

In the logarithmic case, we get instead a t^σ control.

Supercoulomb case via modulated free energy method

In the case of the torus: prove **exponential decay** of the (derivatives of the) limiting solution

Theorem (Chodron de Courcel - Rosenzweig - S '23)

Riesz case $s \in [d-2, d)$, gradient flow with additive noise. We have global in time convergence:

$$\mathcal{F}_N^\beta(f_N^t, \mu^t) \le C \left(\mathcal{F}_N^\beta(f_N^0, \mu^0) + N^{\frac{s}{d}-1} \right).$$

The attractive log case

$$(-\Delta)^{d/2}\mathrm{g} = \delta_0 - 1 \quad \text{on } \mathbb{T}^d$$

$$\mathrm{g}(x) \simeq \log|x|$$

The attractive log case

$$(-\Delta)^{d/2}\mathrm{g} = \delta_0 - 1 \quad \text{on } \mathbb{T}^d$$

$$\mathrm{g}(x) \simeq \log|x|$$

On the torus $\mu_{unif} = 1$ is always a stationary solution of (MF). We show it becomes **unstable** for $\beta > \beta_s$.

The attractive log case

$$(-\Delta)^{d/2} \mathrm{g} = \delta_0 - 1 \quad \text{on } \mathbb{T}^d$$

$$\mathrm{g}(x) \simeq \log |x|$$

On the torus $\mu_{unif} = 1$ is always a stationary solution of (MF). We show it becomes **unstable** for $\beta > \beta_s$.
The modulated energy is no longer coercive. We show that it can be absorbed into the relative entropy if $\beta < \beta_i$:

$$\int F_N(X_N, \mu) df_N(X_N) \leq \frac{1}{\beta} H_N(f_N|\mu^{\otimes N}) + o(1)$$

modulated Log Hardy-Littlewood-Sobolev inequality

Theorem (CdC - R -S)

If $\beta < \beta_i$, *there exists* $\delta_\beta > 0$ *s.t. if* μ *solution to (MF) with* $\|\log \mu^0\|_{L^\infty} \leq \delta_\beta$ *and* f_N^t *an entropy solution to the fwd Kolmogorov eq, then*

$$H_N(f_N^t|(\mu^t)^{\otimes N}) \leq C_{\beta,\|\log \mu^0\|_{W^{2,\infty}}} \mathcal{F}_N^\beta(f_N^0, \mu^0) + CN^{-\gamma}$$

$\rightsquigarrow$ **uniform in time convergence**

Theorem (CdC - R -S)

If $\beta < \beta_i$, there exists $\delta_\beta > 0$ s.t. if μ solution to (MF) with $\|\log \mu^0\|_{L^\infty} \leq \delta_\beta$ and f_N^t an entropy solution to the fwd Kolmogorov eq, then

$$H_N(f_N^t|(\mu^t)^{\otimes N}) \leq C_{\beta,\|\log \mu^0\|_{W^{2,\infty}}} \mathcal{F}_N^\beta(f_N^0, \mu^0) + CN^{-\gamma}$$

$\rightsquigarrow$ **uniform in time convergence**

If $\beta > \beta_s$, for all $\varepsilon > 0$ there exists μ_ε sol to (MF) s.t.

$$\|\mu_\varepsilon^0 - \mu_{unif}\|_{W^{1,\infty}} = O(\varepsilon)$$

and for some $t_\varepsilon \leq C|\log \varepsilon|$

$$\|f_{N,1}^{t_\varepsilon} - \mu_{unif}\|_{L^1} + \|f_{N,1}^{t_\varepsilon} - \mu_\varepsilon^{t_\varepsilon}\|_{L^1} \geq \frac{1}{2}$$

where $f_N^0 = (\mu_\varepsilon^0)^{\otimes N}$

Coro: We cannot have $H_N(f_N^t|(\mu^t)^{\otimes N}) \leq CH_N(f_N^0|\mu_0^{\otimes N}) + o(1)$
(apply ineq to both $\mu = \mu_{unif}$ and μ_ε^0)
no uniform in time convergence

Proof of the commutator estimate: the electric rewriting of the energy

Set $h^f = \mathrm{g} * f$. In the Coulomb case

$$-\Delta h^f = c_d f$$

We have by IBP

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{g}(x-y)df(x)df(y) = \int_{\mathbb{R}^d} h^f df = -\frac{1}{c_d}\int_{\mathbb{R}^d} h^f \Delta h^f = \frac{1}{c_d}\int_{\mathbb{R}^d} |\nabla h^f|^2.$$

Proof of the commutator estimate: the electric rewriting of the energy

Set $h^f = \mathrm{g} * f$. In the Coulomb case

$$-\Delta h^f = c_d f$$

We have by IBP

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{g}(x-y) df(x) df(y) = \int_{\mathbb{R}^d} h^f df = -\frac{1}{c_d} \int_{\mathbb{R}^d} h^f \Delta h^f = \frac{1}{c_d} \int_{\mathbb{R}^d} |\nabla h^f|^2.$$

Positivity of F_N not clear! Use suitable **truncations** obtained by replacing δ_{x_i} by $\delta_{x_i}^{(r_i)}$ with $r_i =$ nearest neighbor distance. Use almost-**monotonicity** with respect to truncation parameter.

Proof in the Coulomb case

Stress-energy tensor

$$[\nabla h^f]_{ij} = 2\partial_i h^f \partial_j h^f - |\nabla h^f|^2 \delta_{ij}.$$

For regular f,

$$\mathrm{div}\ [\nabla h^f] = 2\Delta h^f \nabla h^f = -\frac{2}{c_d} f \nabla h^f.$$

Proof in the Coulomb case

Stress-energy tensor

$$[\nabla h^f]_{ij} = 2\partial_i h^f \partial_j h^f - |\nabla h^f|^2 \delta_{ij}.$$

For regular f,

$$\mathrm{div}\, [\nabla h^f] = 2\Delta h^f \nabla h^f = -\frac{2}{c_d} f \nabla h^f.$$

$$\begin{aligned} &\iint_{\mathbb{R}^d\times\mathbb{R}^d} (\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y)df(x)df(y) \\ &\quad = 2\int_{\mathbb{R}^d} \psi(x)\cdot \nabla h^f(x)df(x) = -c_d \int_{\mathbb{R}^d} \psi\cdot \mathrm{div}\, [\nabla h^f] \\ &\quad = c_d \int_{\mathbb{R}^d} D\psi\cdot[\nabla h^f] \le c_d \|D\psi\|_{L^\infty} \iint \mathrm{g}(x-y)df(x)df(y) \end{aligned}$$

Then needs to be **renormalized** by truncation procedure.

Proof in the Coulomb case

Stress-energy tensor

$$[\nabla h^f]_{ij} = 2\partial_i h^f \partial_j h^f - |\nabla h^f|^2 \delta_{ij}.$$

For regular f,

$$\mathrm{div}\ [\nabla h^f] = 2\Delta h^f \nabla h^f = -\frac{2}{c_d} f \nabla h^f.$$

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\psi(x) - \psi(y)) \cdot \nabla \mathrm{g}(x-y) df(x) df(y) \\
&\quad = 2 \int_{\mathbb{R}^d} \psi(x) \cdot \nabla h^f(x) df(x) = -c_d \int_{\mathbb{R}^d} \psi \cdot \mathrm{div}\ [\nabla h^f] \\
&\quad = c_d \int_{\mathbb{R}^d} D\psi \cdot [\nabla h^f] \le c_d \|D\psi\|_{L^\infty} \iint \mathrm{g}(x-y) df(x) df(y)
\end{aligned}$$

Then needs to be **renormalized** by truncation procedure.
[Nguyen-Rosenzweig-S]: dispense with stress-tensor structure and from paraproduct commutator estimates. Purely space-based. (Uses Riesz transform estimates).

THANK YOU FOR YOUR ATTENTION!