Non-ergodic dynamics induced by measurements

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INHOMOGENEOUS RANDOM SYSTEMS
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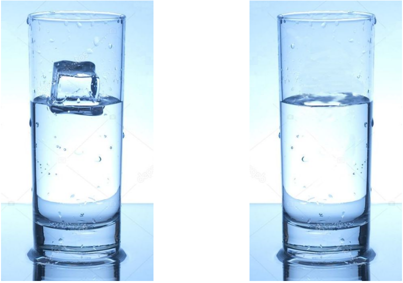
OUTLINE

- Recap on thermalisation in many-body quantum systems
- Entanglement production:
  - membrane picture
  - random unitary circuits
- Monitored systems and measurement-induced phase transition
- Classically monitored systems
  - mapping to disordered systems
  - directed polymer solution
- Back to quantum
  - purification dynamics
  - random matrices
  - universality

Out-of-equilibrium dynamics of isolated many-body quantum systems

Fundamental questions

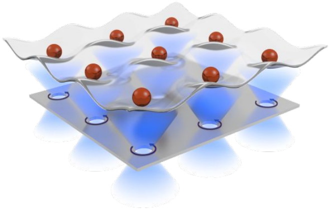
- How does a many-body system thermalise?
- Universality behind thermalization?
- Can thermalization be avoided?
- **New out-of-equilibrium phases?**

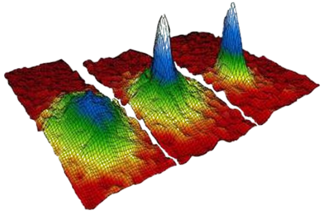


Thermalization = loss of memory of initial conditions

Experimental progress

cold atoms, trapped ions, etc. → fine-tuned interaction in isolated many-body quantum systems



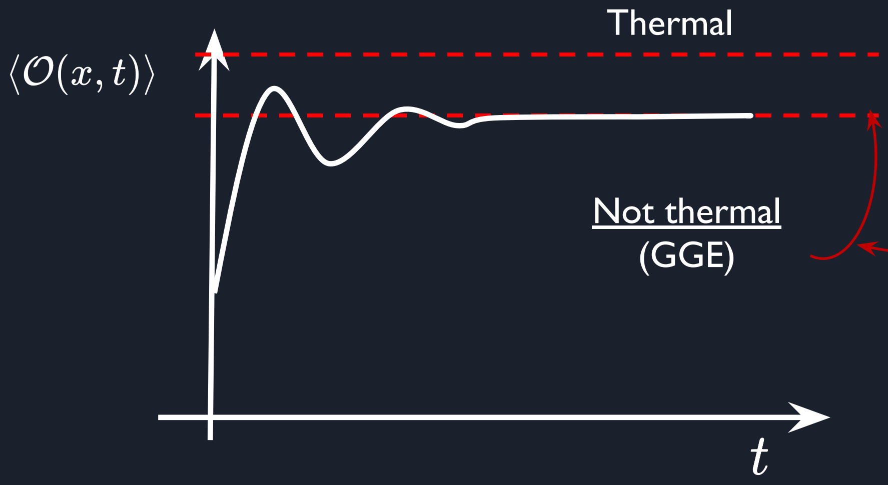


Sudden quantum quenches in homogeneous systems

1. Initial homogeneous high-energy state
2. Evolution with homogeneous H
3. Local relaxation to a steady state

CFT,
Calabrese, Cardy, '06

Emergent statistical description?



Fix by conserved quantities

$$\hat{\mathcal{Q}} = \int dx\, \hat{q}(x) \;,\; [\hat{H}, \hat{\mathcal{Q}}] = 0$$

$$\langle \psi(t) | \hat{q}(x) | \psi(t) \rangle = \langle \psi(0) | \hat{q}(x) | \psi(0) \rangle$$

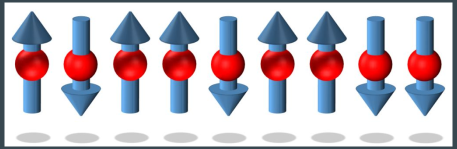
enforce conservation of charges

$$\rho \propto e^{-\sum_j \beta_j \mathcal{Q}_j}$$

Rigol *et al* '07

Role of Locality

1. Only way to observe relaxation is to use a smaller class of observables
2. A many-body system is extended in space

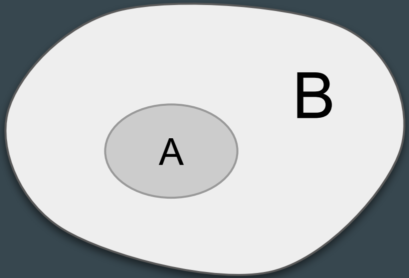


$$\mathcal{H} = \otimes_{i=1}^{L} \mathcal{H}_i$$

3. Can the system behave as its own "bath"?

Local observables (with support only in A) can relax:

$$\langle O(x) \rangle_\infty = \lim_{t \to \infty} \langle O(x,t) \rangle$$



Thermalisation and entanglement

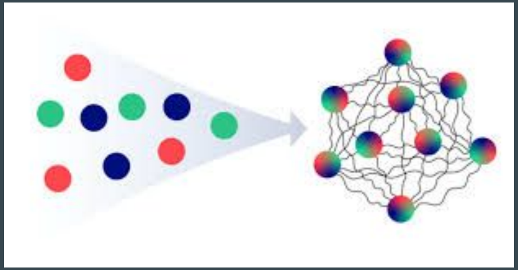
- Entanglement is a distinctive and unique feature of quantum mechanics

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

- It implies information is partially lost when a portion of the system is discarded

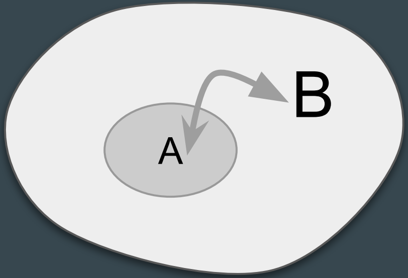
$$\rho = |\psi\rangle\langle\psi| \,, \qquad \rho_A = \mathrm{Tr}_B[\rho] \,, \qquad \mathrm{Tr}[\rho_A^2] < 1$$

- Thermalization must produce a lot of entanglement



$$\rho_A \sim \frac{e^{-\beta H_A}}{Z}$$

$$\mathrm{Tr}[\rho_A^2] \sim e^{-s_2 L_A}$$



WHAT HAS BEEN DONE

Standard methods do not apply: far from the groundstate, no small parameter

Numerical methods

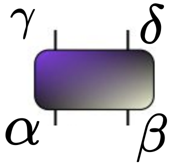
- DMRG, exact diagonalization
- Restrictions: small times, small sizes

Integrable models

- Bethe-Ansatz, free theories
- Restrictions: fine-tuned, non-ergodic dynamics
- standard ETH does not hold

RANDOM CIRCUITS

Hilbert space of
a single q-dit

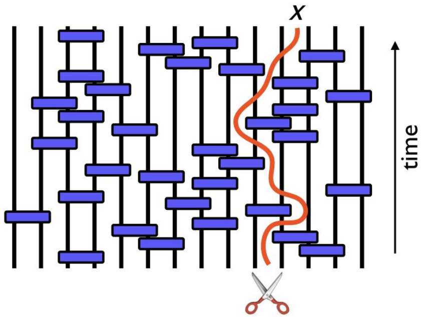


$$= U_{\alpha,\beta;\gamma,\delta}$$

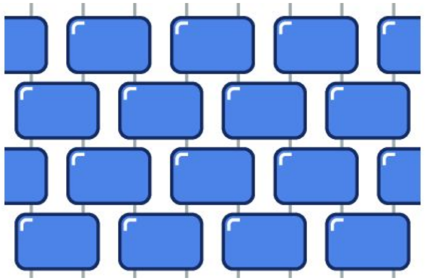
gate acting on two sites

Recipe for building random circuits

- consider many q-dits
- choose a geometry to your liking
- make the q-dits interact with random gates (Haar distributed)



Skinner et al. PRX 7, 031016 (2017)

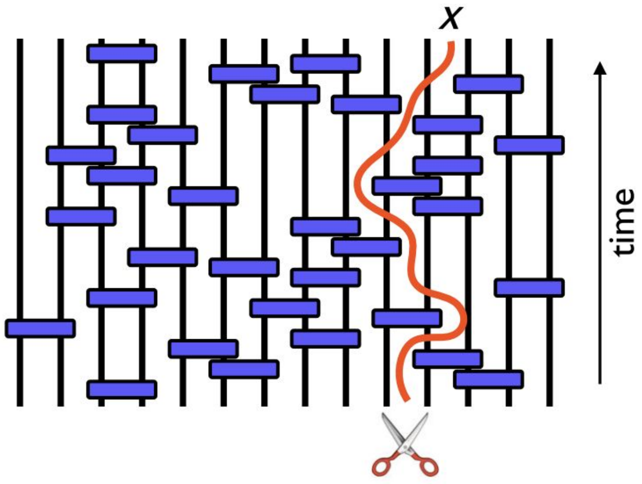


ENTANGLEMENT GROWTH & MINIMAL CUT

Renyi entropies: $S_n(x) = \frac{1}{1-n}\log(\mathrm{Tr}[\rho_x^n])$

Von Neumann: $S_{VN}(x) = \lim_{n\to 1} S_n(x) = -\mathrm{Tr}[\rho_x \log(\rho_x)]$

Hartley entropy: $S_0(x)$ Number of non-vanishing eigenvalues



$S_0(x) \propto$ length of the minimal cut $\sim t$

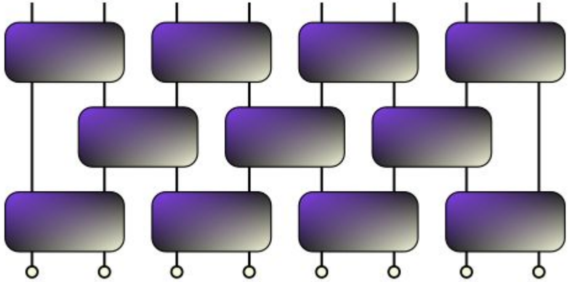
Skinner et al. PRX 7, 031016 (2017)

In general, entanglement entropies grow linearly with t

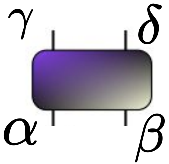
Random Unitary circuits

- Consider local degrees of freedom (e.g. spins)
- Evolve with local interactions

$$|\Psi(t)\rangle = W|\Psi(0)\rangle =$$



- Each gate is independently chosen as a (small) random unitary matrix

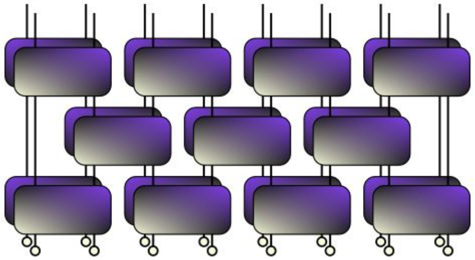


$$= U_{\alpha,\beta;\gamma,\delta}$$

- No conserved quantity is present (not even the energy!)
- **Diagrammatic notation is advantageous**

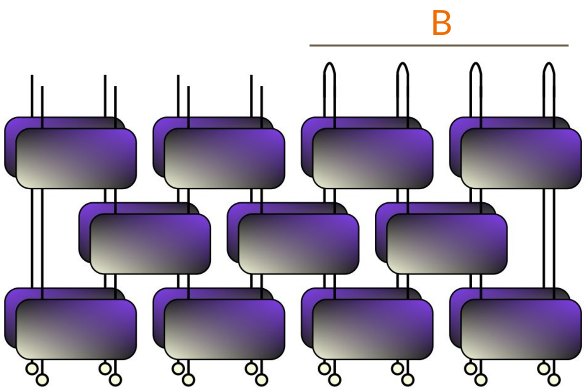
Example: expressing the purity

$$|\Psi(t)\rangle\langle\Psi(t)| =$$

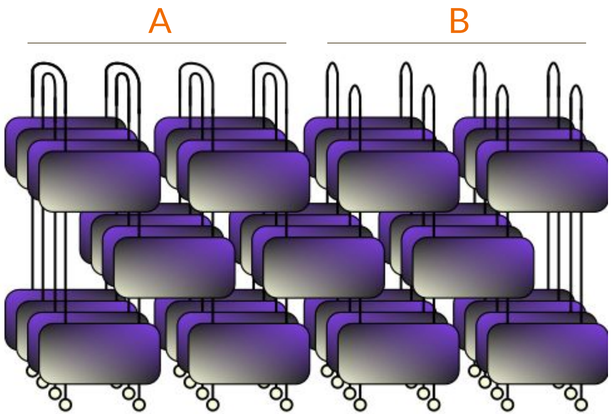


$$\rho_A = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|]\ , \mathcal{P} = \mathrm{Tr}[\rho_A^2]$$

$$\rho_A =$$

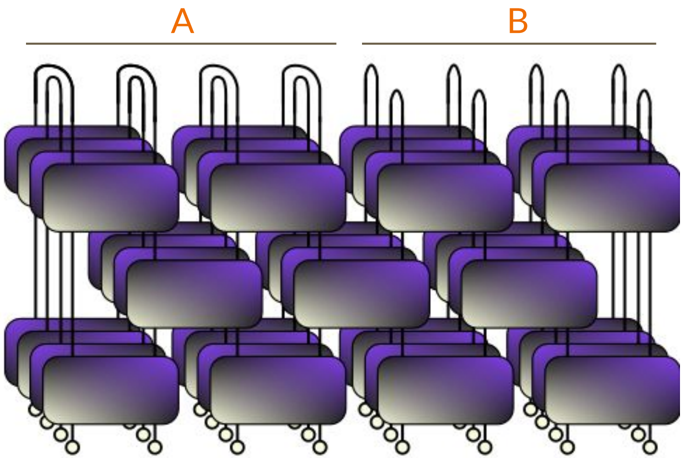


$$\mathcal{P} =$$

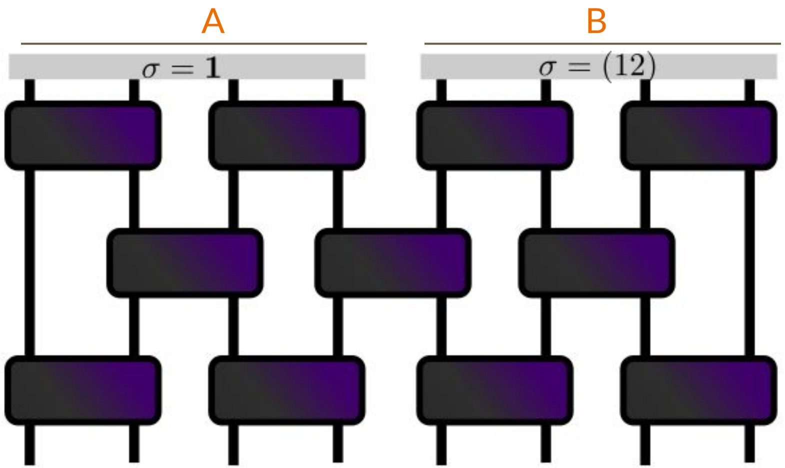


Purity as a classical partition function

$\mathcal{P} =$

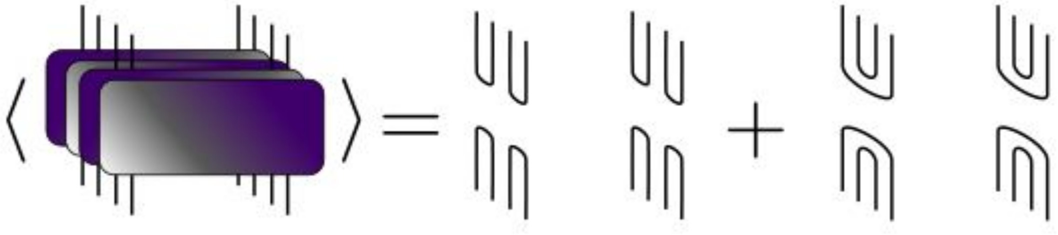


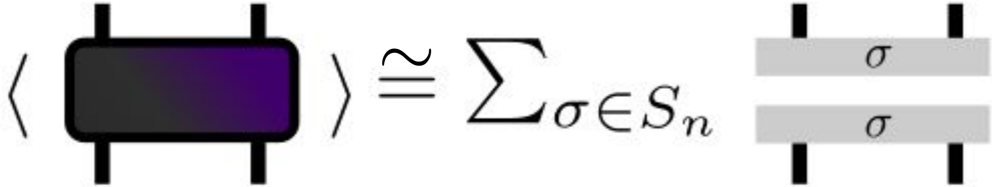
$=$



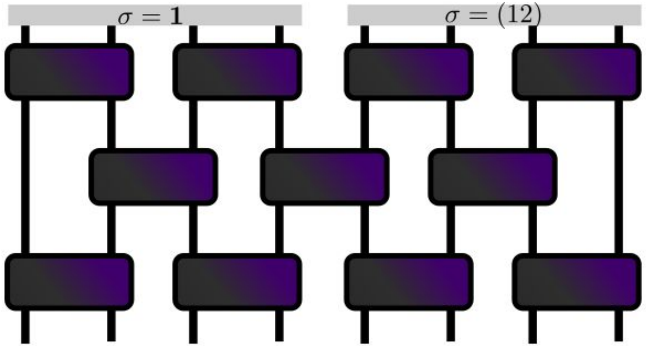
Diagrammatic average for random matrix

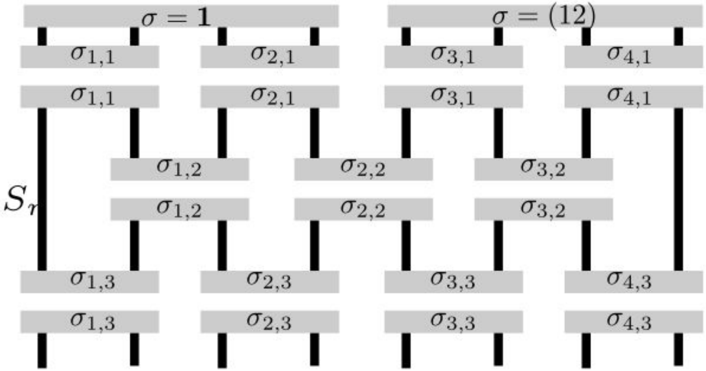
- average of large unitary matrices are done similarly





Average purity as a classical partition function

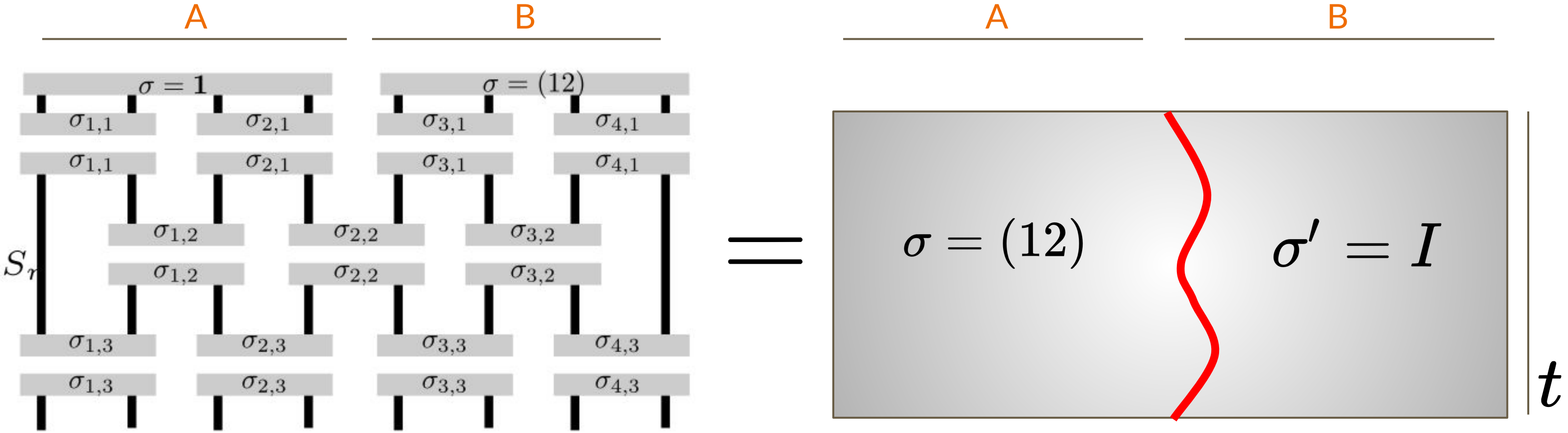
$\langle \mathcal{P} \rangle = \Big\langle$ 

$\Big\rangle = \sum_{\{\sigma\} \in S_r}$ 

Classical partition function where local degrees of freedom range over permutations

Coarse-grained / membrane picture

$\langle \mathcal{P} \rangle = \sum_{\{\sigma\} \in S_n}$



T Zhou, A Nahum - PRX, 2020, A Nahum et al Phys. Rev. X 7,

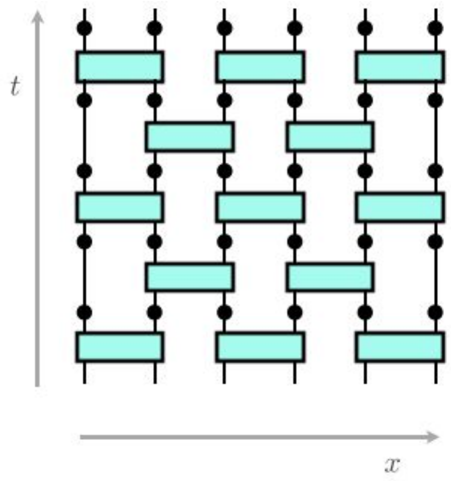
$\langle \mathcal{P} \rangle \sim e^{-s_2 t}$, free energy of a membrane extended in time

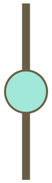
Beyond unitary dynamics: including measurements

UNITARY

+

UNITARY DYNAMICS + MEASUREMENTS





perform a projective measurement of S_z with probability p

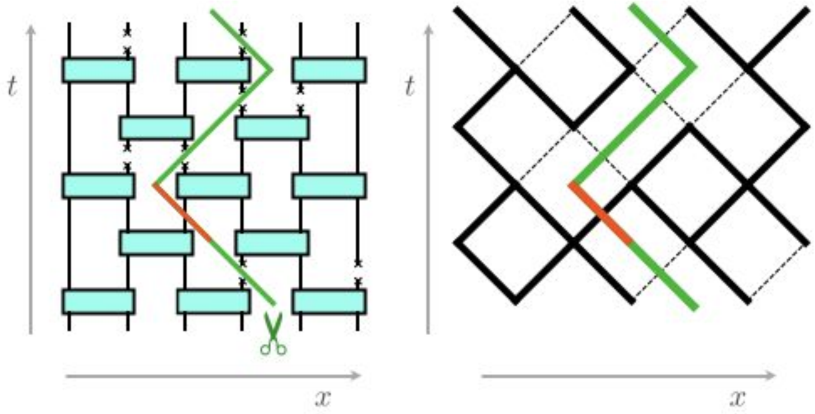
$$|\Psi_0\rangle \to |\Psi_t(\mathbf{a} = \text{positions + outcomes of measurements})\rangle$$

what is the entanglement of the resulting quantum state?

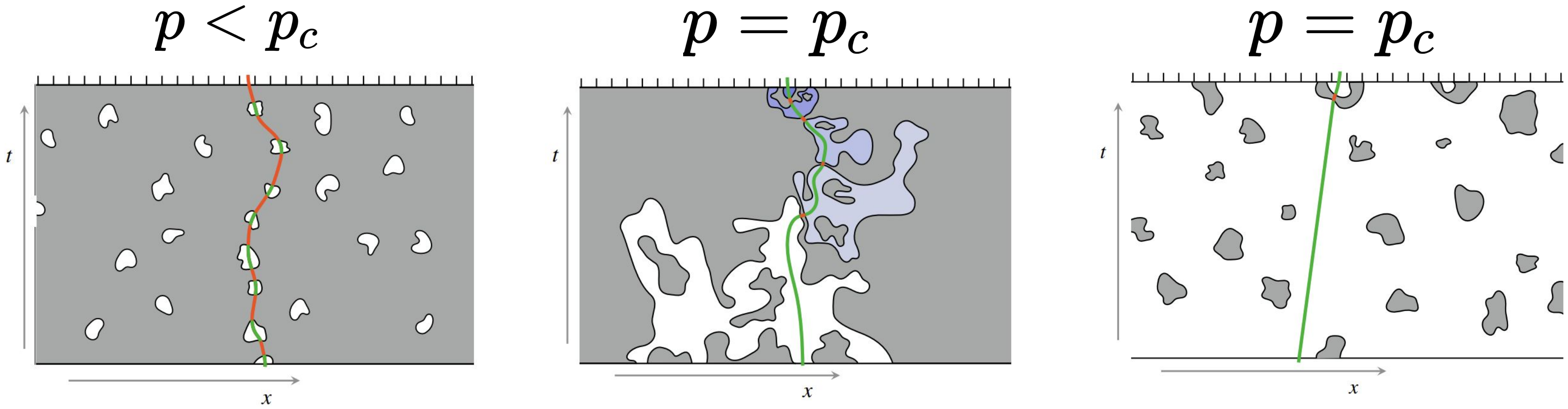
B. Skinner *et al.*, PRX, 9 (2019)

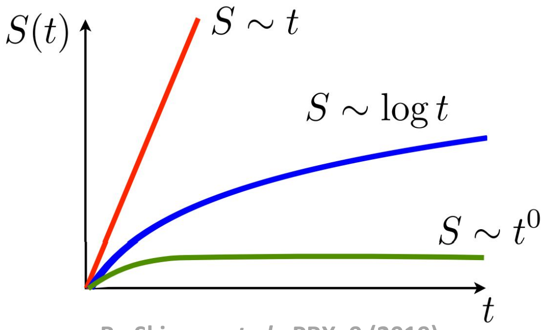
FROM MINIMAL CUT TO DIRECTED PERCOLATION

BONDS WHERE MEASUREMENTS OCCUR HAVE NO “COST” FOR THE PATH



COARSE-GRAINED PICTURE





B. Skinner *et al.*, PRX, 9 (2019)

MEASUREMENT-INDUCED PHASE TRANSITIONS (MIPT)

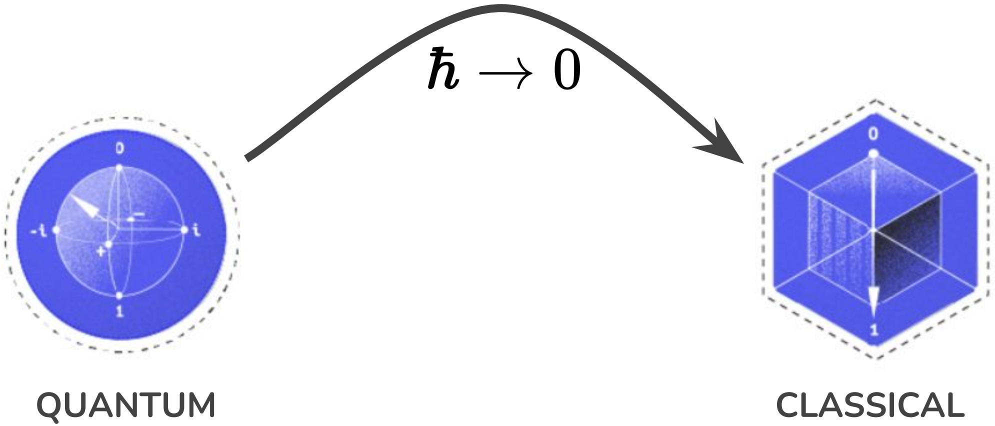
volume law entangled phase | p_c | area law disentangled phase

- Hartley entropy S_0 / q→∞: MIPT = classical directed percolation, **NOT TRUE IN GENERAL**

- Essentially impossible to observe (post-selection)
- Simulatable vs non-simulatable phase

- Hard to study analytically in general
  - it requires strong interactions (absent in non-interacting theories)
  - intrinsically stochastic

Spinoff: Explore the classical analogous of MIPT



Classical Markov chain

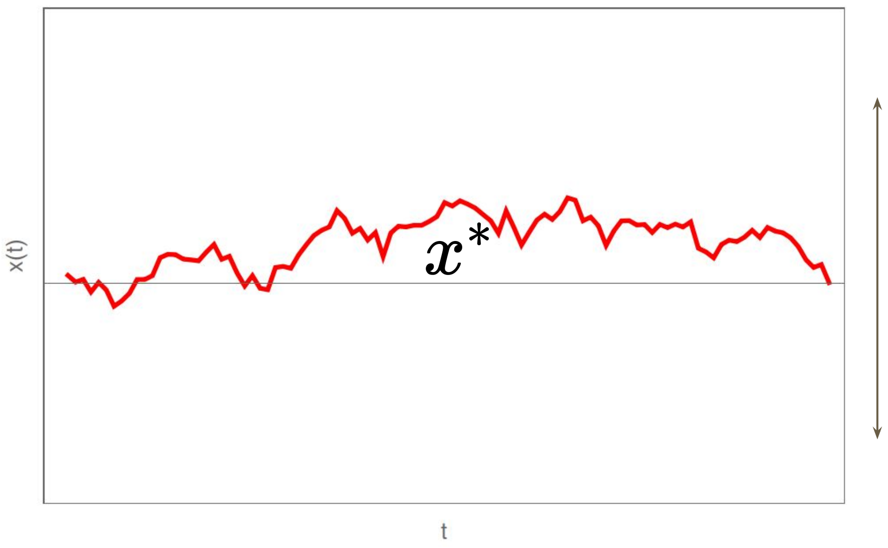
- system following stochastic dynamics

$$dx = f(x)dt + g(x)dW$$

- an observer tries to locate x

$$\text{Fokker-Planck} \rightarrow p(x)$$

e.g. random walk



Shannon entropy: $S[p] = -\int dx\, p(x,t) \log p(x,t)$

uncertainty grows with time

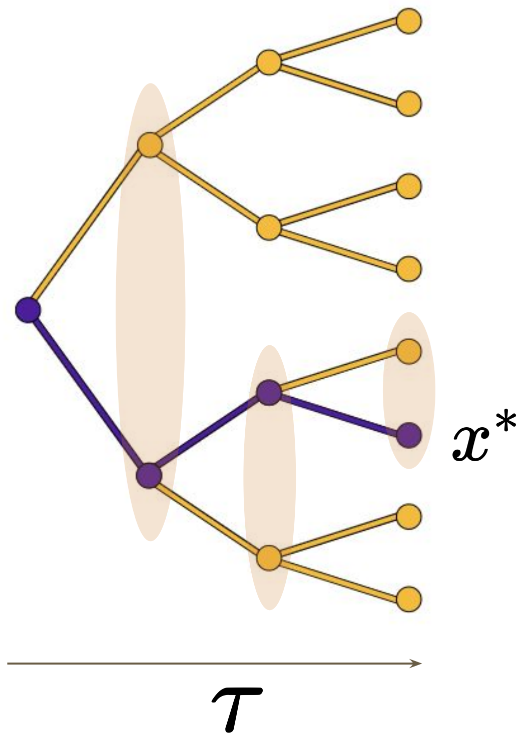
Example: directed random walk on a tree

$x^* \ =$ a random path on the tree

Motivation

- solvability
- interpretation as Lyapunov exponent of chaotic system

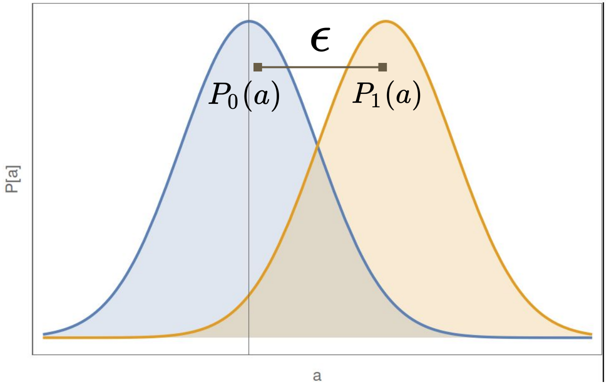
arXiv: 2501.00547
see also: SWP Kim, A Lamacraft (2404.07263)



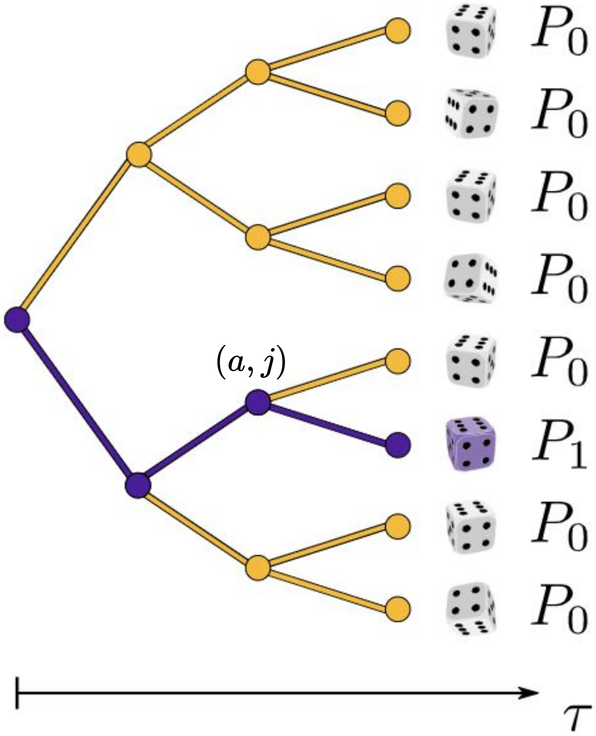
Mitigate uncertainty with measurements

- an observer performs measurements on each state at each time

$$a_j \sim \mathrm{Prob}(a_j|x) = \begin{cases} P_0(a_j) & j \text{ is empty} \\ P_1(a_j) & j \text{ is full} \end{cases}$$



ϵ = signal to noise ratio



Mitigate uncertainty with measurements

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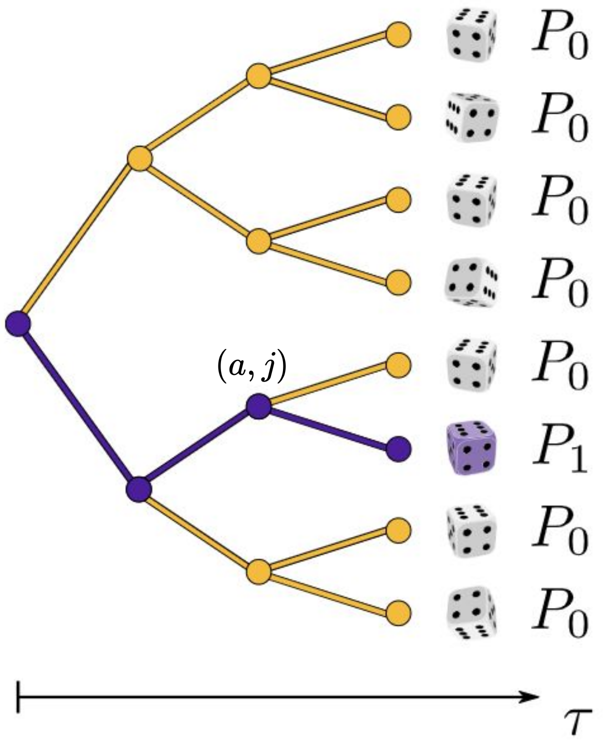
$$a_j \sim \text{Prob}(a_j|x) = \begin{cases} P_0(a_j) & j \text{ is empty} \\ P_1(a_j) & j \text{ is full} \end{cases}$$

- use measurements to reconstruct the distribution (Bayes's theorem)

$$\text{P}(x|\mathbf{a}) \propto \frac{\text{Prob}(a|x)}{\sum_{x'} \text{Prob}(a|x')}$$

$$P(\mathbf{a}) = P(\mathbf{a}|x^*)$$



Mapping to directed polymer on the Cayley tree

- disorder on each node chosen according

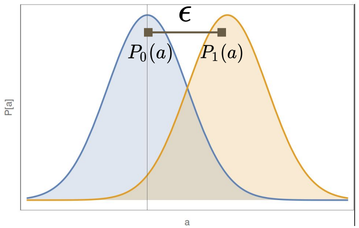
$$a_j \sim P_0(a_j)$$

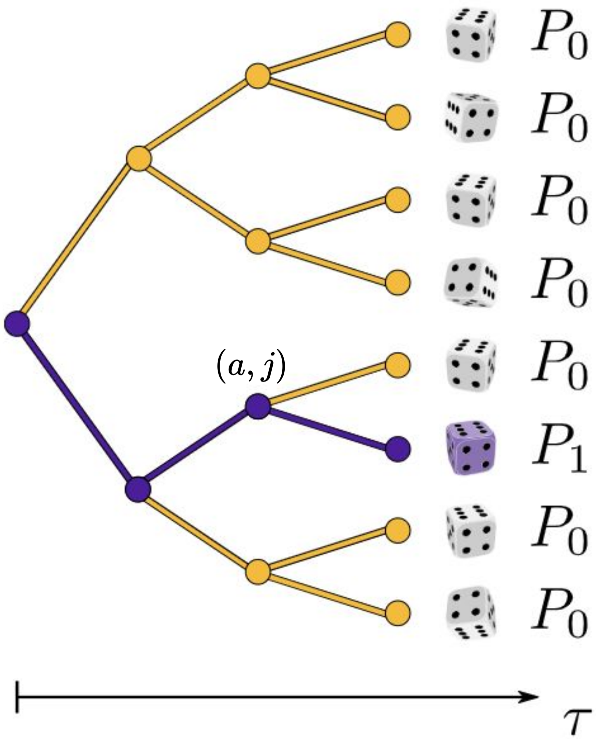
- Boltzmann weight of a given path

$$z_x = \prod_{j \in x} \frac{P_1(a_j)}{P_0(a_j)}$$

$\epsilon \gg 1 \Rightarrow$ strong disorder / low temperature phase

$\epsilon \ll 1 \Rightarrow$ weak disorder/ high temperature phase



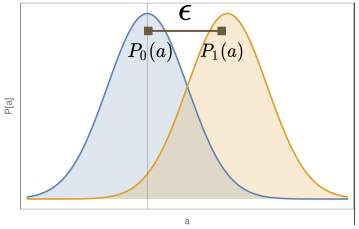


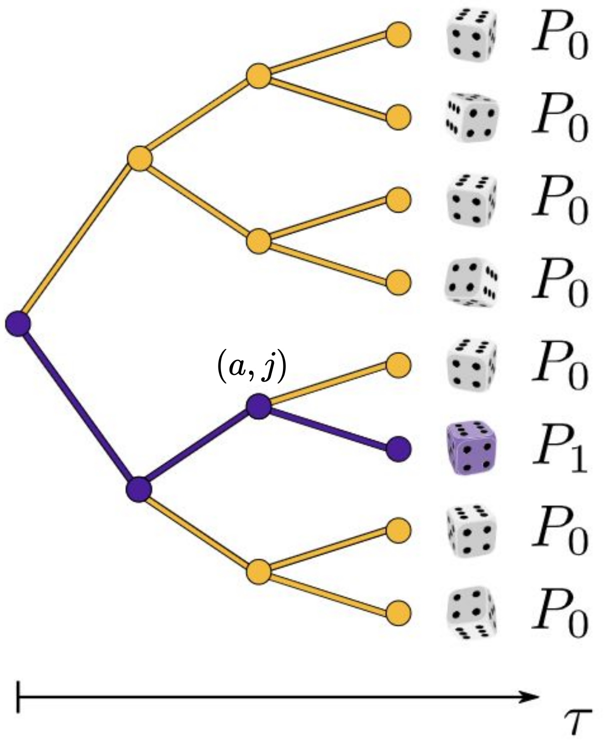
Mapping to directed polymer on the Cayley tree

INFERENCE FROM A TIME SERIES

$\epsilon \gg 1 \Rightarrow$ strong disorder / low temperature phase

$\epsilon \ll 1 \Rightarrow$ weak disorder/ high temperature phase



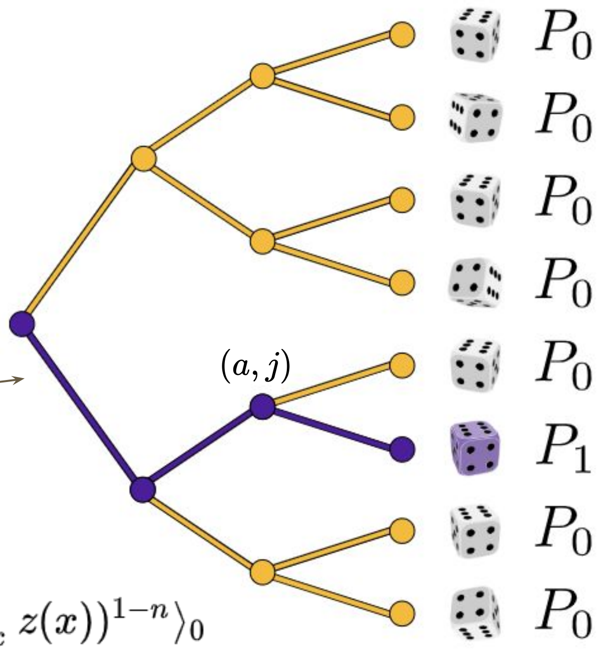


How to estimate whether reconstruction is possible?

$$p(x) = \text{inferred probability} = \text{Prob}(x|\mathbf{a})$$

$$\sum_x p(x) = 1 \Leftarrow \text{ normalisation}$$

narrow distribution implies measurements are effective



reference trajectory

$$\langle F[p(x)]\rangle \equiv \left\langle F\left[\frac{z(x)}{\sum_{x'} z(x')}\right]\left(\sum_{x^*} z(x^*)\right)\right\rangle_0$$

e.g. participation ratio $\langle \sum_x p(x)^n \rangle = \langle \sum_x z(x)^n \left(\sum_x z(x)\right)^{1-n} \rangle_0$

unusual replica limit compared to standard disordered systems

Few words about directed polymer on the tree

$$Z := \sum_{x\in\text{path}} Z(x) = \sum_{x\in\text{path}} \prod_{j\in x} \frac{P_1(a_j)}{P_0(a_j)} = \sum_{x\in\text{path}} \prod_{j\in x} B(a_j)$$

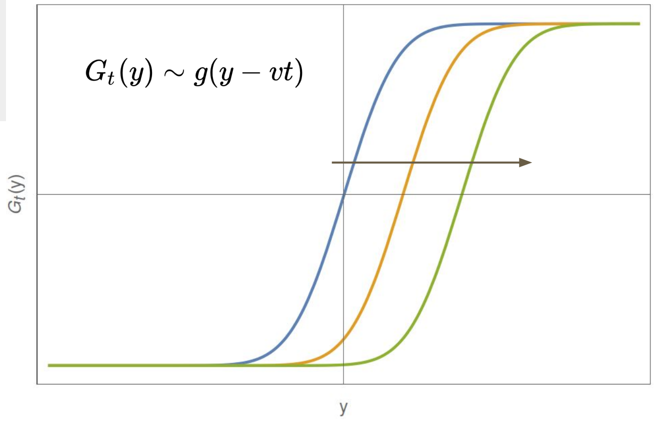
Derrida&Spohn: $G_t(y) = \langle e^{-e^{-y}Z_t} \rangle_0 \quad + \quad$ recursive relation

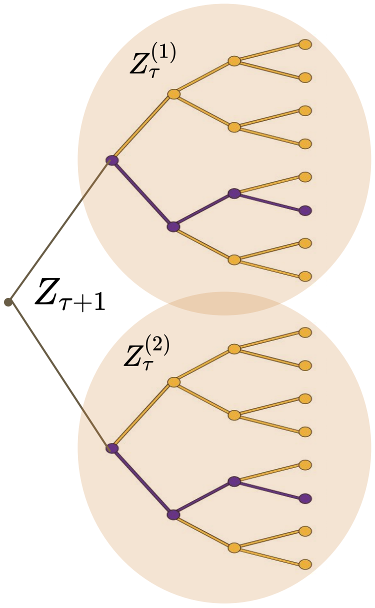
$$G_{\tau+1}(y) = \left\langle G_\tau\left(y - \ln B(a)\right)^K \right\rangle_0 .$$

reaction-diffusion equation morally like KPP

$$\partial_t H = D\partial_x^2 H + \lambda H(1-H)$$
$$H = 1 - G$$

$$\log Z \sim vt$$





Shannon entropy of the estimated distribution

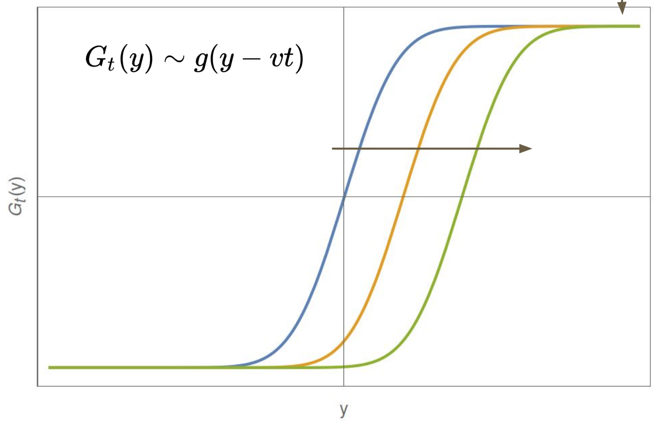
Shannon entropy: $S[p] = -\int dx\, p(x,t) \log p(x,t) \quad \Leftarrow \quad F[p] := -p \log p$

$$\langle S[p] \rangle = \int dy e^{y} (G_t(y) - G_0(y))$$

$$G_t(y) \overset{y\to\infty}{\sim} 1 - e^{-y} + O(e^{-2y})$$

Morally "$Z \log Z$" instead of "$\log Z$"

behavior of entropy controlled by faster-than-front



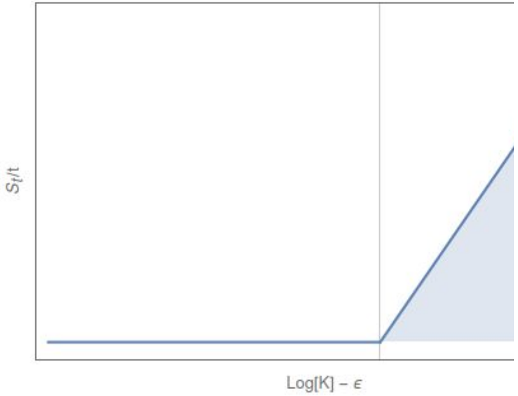
Regime of atypical events:

$$u_t(y) = e^{y}(1 - G_t(y))$$

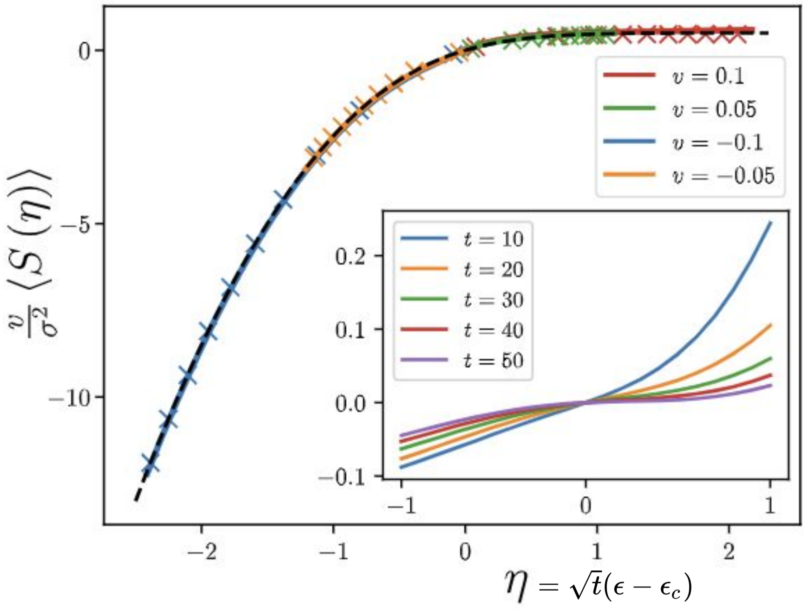
KPP reduces to diffusion in the presence of a hard wall

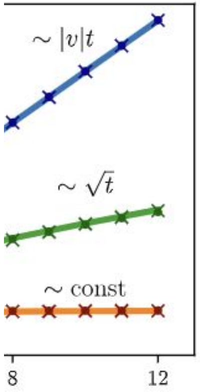
Phase transition in the rate of entropy production

Shannon entropy: $S[p] = -\int dx\; p(x,t) \log p(x,t) \quad \Leftarrow \quad F[p] := -p \log p$



$$\mathcal{S}(\eta) = \left(\frac{1}{2} + \eta^2\right) \operatorname{erf} \eta - \eta^2 + \frac{\eta}{\sqrt{\pi}} e^{-\eta^2} \; .$$





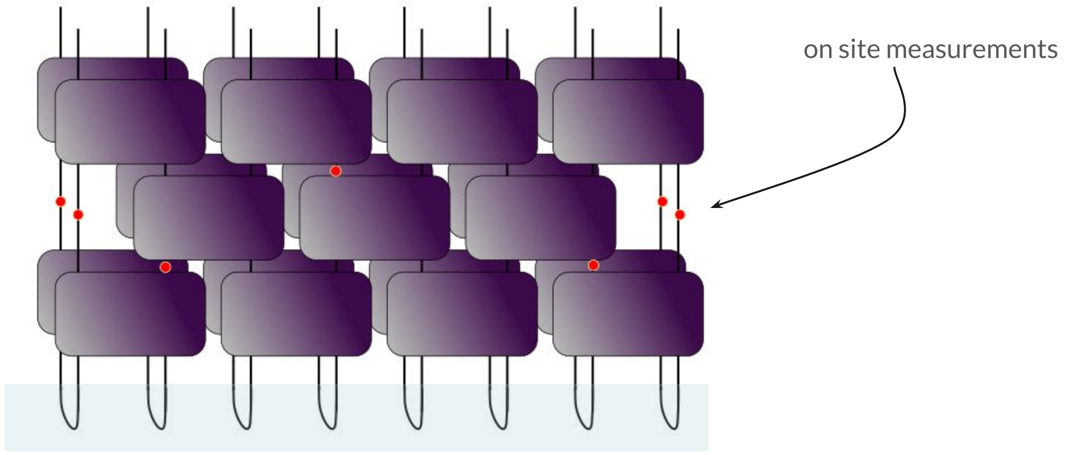
back to quantum

$$\hbar \neq 0$$

FROM BAYES TO BORN'S RULE

$$p(x|a) \longrightarrow \rho_a \ , \quad \text{density matrix}$$

$\tilde{\rho}_a =$



tunable parameter: probability of measuring each site p

Normalisation + Born's rule: $\rho_a = \frac{\tilde{\rho}_a}{\mathrm{Tr}[\tilde{\rho}_a]}$ with probability $\mathrm{Tr}[\tilde{\rho}_a]$

REPLICA TRICK

Normalisation + Born's rule: $\rho_a = \frac{\tilde{\rho}_a}{\mathrm{Tr}[\tilde{\rho}_a]}$ with probability $\mathrm{Tr}[\tilde{\rho}_a]$

$$\langle F[\rho_{\mathbf{a}}]\rangle_{\text{meas}} = \left\langle F\left[\frac{\tilde{\rho}_{\mathbf{a}}}{\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]}\right]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]\right\rangle_{\text{unbiased}}$$

Computation of the purity

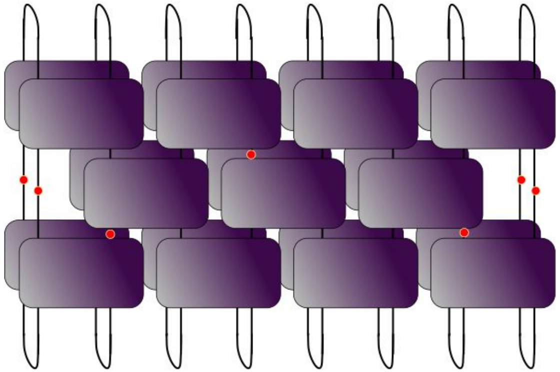
$$\text{e.g. } F[\rho] = \mathrm{Tr}[\rho^2]$$

$$\langle \mathrm{Tr}[\rho_{\mathbf{a}}^2]\rangle_{\text{meas}} = \lim_{N\to 1}\left\langle \mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}^2]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]^{N-2}\right\rangle_{\text{unbiased}}$$

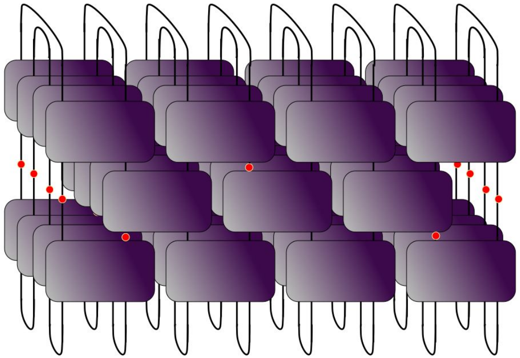
again unusual replica limit

DIAGRAMS AND PERMUTATIONS

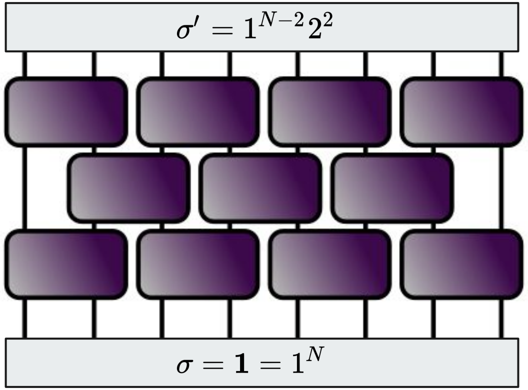
$\mathrm{Tr}[\tilde{\rho}_a] =$



$\mathrm{Tr}[\tilde{\rho}_a^2] =$



$\lim_{N\to 1} \mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}^2]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]^{N-2} =$



COMPETITION BETWEEN ORDER / DISORDER

$$p_c$$

UNITARY DYNAMICS DOMINATES

$$\mathrm{Tr}[\rho^{a_1}]\mathrm{Tr}[\rho^{a_2}]\mathrm{Tr}[\rho^{a_3}]\dots$$
$$= \mathrm{Tr}[\Omega\rho^{\otimes n}]$$

EXACTLY PRESERVED IN TIME
(TENDENCY TO ORDER)

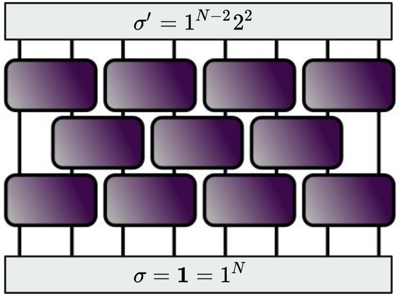
MEASUREMENTS DOMINATE

$$\mathrm{Tr}[\rho^a] \to 1$$

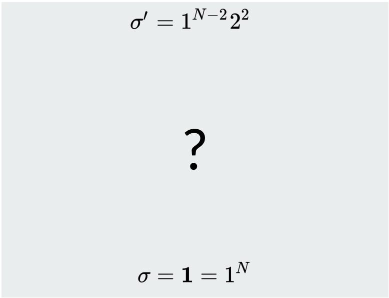
DYNAMICS PURIFIES EVERY STATE
(TENDENCY TO DISORDER)

PURIFICATION TIME AND SYMMETRY BREAKING

$$\mathrm{Tr}[\rho_{\mathbf{a}}^2] \equiv \lim_{N\to 1} \mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}^2]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]^{N-2} =$$

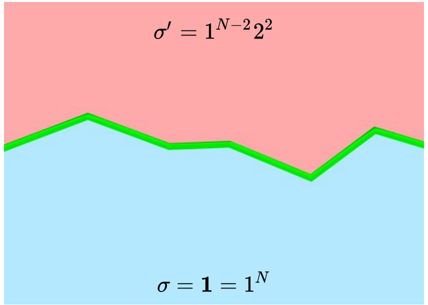


$=$



SYMMETRY BROKEN PHASE (weak measurements)

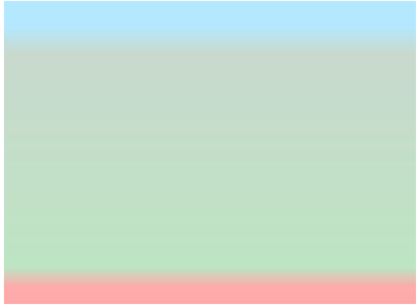
$\mathrm{Tr}[\rho_{\mathbf{a}}^2] =$



$\sim te^{-\alpha N}$

UNBROKEN PHASE (strong measurements)

$\mathrm{Tr}[\rho_{\mathbf{a}}^2] =$



$= O(1)$

PURIFICATION TIME

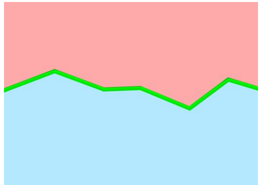
volume law
entangled phase

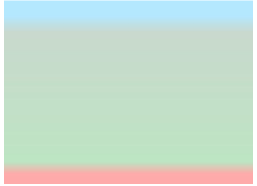
p_c

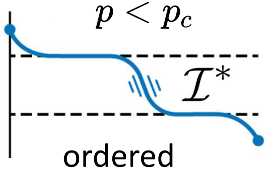
area law
disentangled phase

Prepare the system in the fully mixed state

$$\rho(t=0) = \mathbf{I}/2^N$$



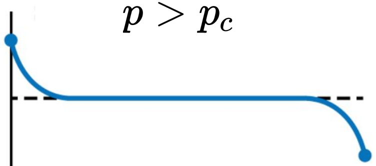




State eventually becomes pure

$$S[\rho] \overset{t=0}{\sim} N \log 2\ , \qquad S[\rho] \overset{t\to\infty}{\longrightarrow} 0$$

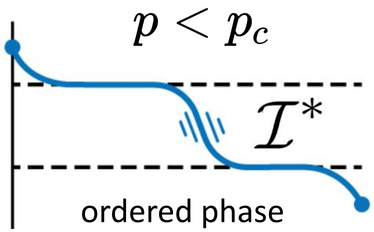
$$\tau_P(p) = ?$$



$$\tau_p(p < p_c) \propto e^{NI}$$

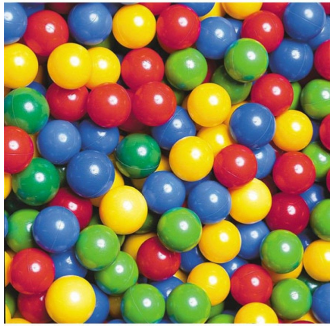
$$\tau_p(p > p_c) = O(1)$$

SLOW PURIFICATION $\longrightarrow$ RMT

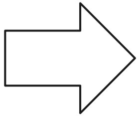


$$\Pi \sim t e^{-L\mathcal{I}^*} = t/\tau_P$$

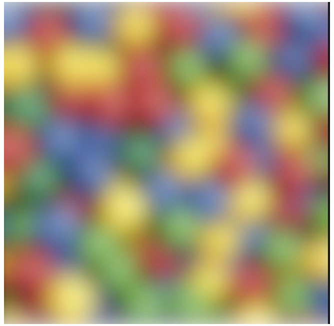
Purification is so slow that we can treat the system as a single dot



$$t/\tau_P = x$$



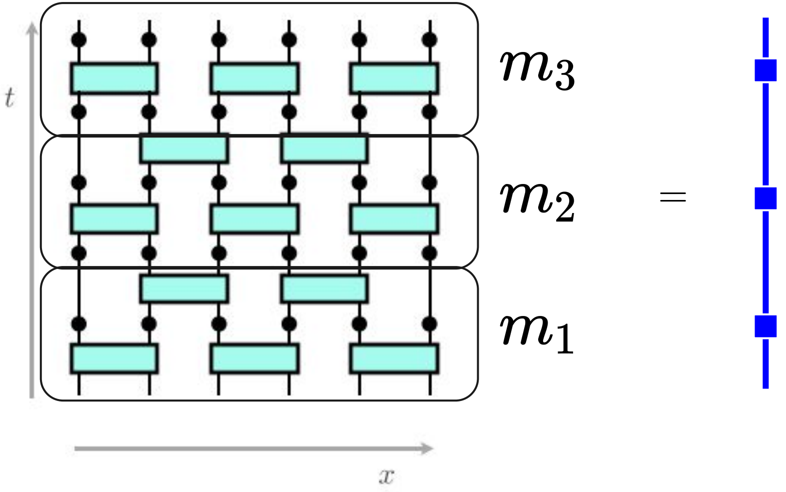
[...] there must come a point where such analyses of individuals level cannot usefully go [...] F. Dyson



remove spatial structure: effective model in 0+1 dimension

ADL, C Liu, A Nahum, T Zhou - arXiv:2312.17744, 2023
F Gerbino, P Le Doussal, G Giachetti, ADL - Quantum Reports, 2024
see also VB Bulchandani, SL Sondhi, JT Chalker, JSTAT 191 (5), 55

SLOW PURIFICATION $\longrightarrow$ RMT



$$\tilde{\rho}_{\mathbf{a}} = m_T \dots m_2 m_1 m_1^\dagger m_2^\dagger \dots m_T^\dagger$$

$$m_i = \begin{pmatrix} x_{11} & x_{12} & \cdots \\ \vdots & \ddots & \end{pmatrix} \sim 2^L \times 2^L$$

Gaussian matrices for simplicity – **Ginibre ensemble**
(Not too important as long as rotational invariant)

Continuous time version also possible (see F Gerbino, P Le Doussal, G Giachetti, ADL - Quantum Reports, 2024)

REPHRASING OF THE PROBLEM WITHIN RMT

$$\tilde{\rho}_{\mathbf{a}} = m_T \dots m_2 m_1 m_1^\dagger m_2^\dagger \dots m_T^\dagger = MM^\dagger \ , \qquad M = m_T \dots m_2 m_1$$

- distribution of singular values of a product of many random matrices
- scaling limit where: matrices are large and many matrices are multiplied

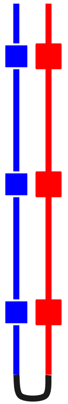
$$\mathcal{N} \to \infty, T \to \infty \ , \quad T/\mathcal{N} = T/\tau_P = x$$

scaling variable

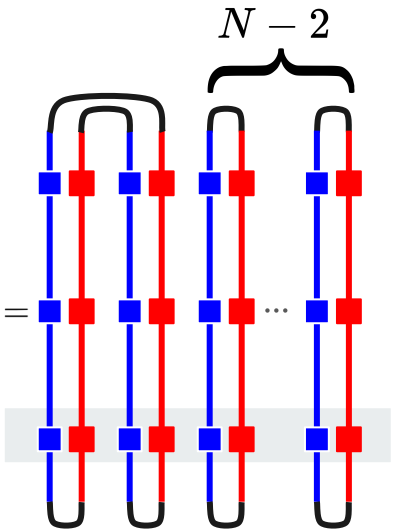
see also D.-Z. Liu, D. Wang, and Y. Wang, arxiv:1810.00433
G. Akemann, Z. Burda, and M. Kieburg, PRE 102, 052134 (2020).

APPROACH VIA REPLICAS

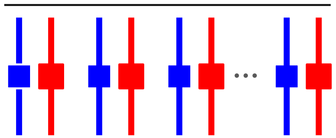
$$\tilde{\rho}_{\mathbf{a}} = m_T \ldots m_2 m_1 m_1^\dagger m_2^\dagger \ldots m_T^\dagger =$$



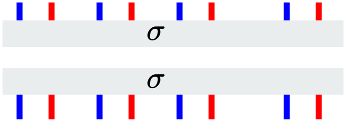
$$\mathrm{Tr}[\rho_{\mathbf{a}}^2] \equiv \lim_{N\to 1} \mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}^2]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]^{N-2} =$$



Wick's theorem:

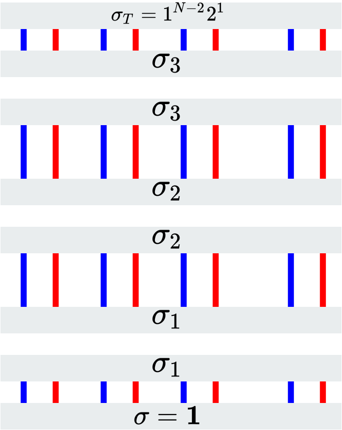


$$= \sum_{\sigma \in S_N}$$



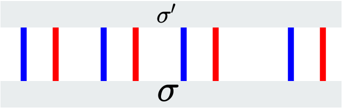
COUNTING PATHS IN THE PERMUTATION GROUP

$$\mathrm{Tr}[\rho_{\mathbf{a}}^2] \equiv \lim_{N\to 1} \mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}^2]\mathrm{Tr}[\tilde{\rho}_{\mathbf{a}}]^{N-2} = \sum_{\sigma_1,\ldots,\sigma_T \in S_N}$$



$$= [T]^t_{\sigma_0,\sigma_T}$$

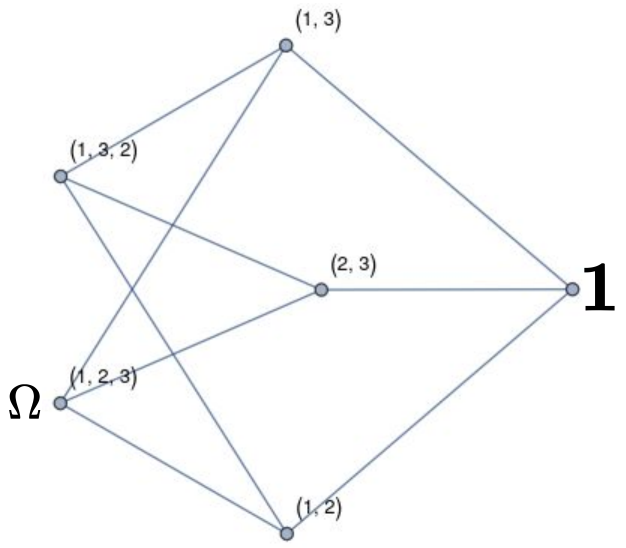
$$T_{\sigma,\sigma'} =$$



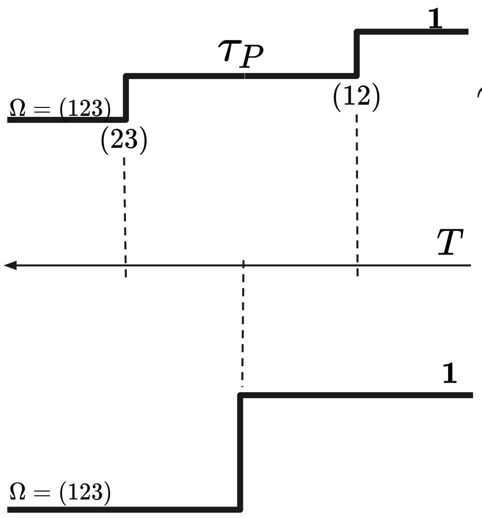
$$= \text{transfer matrix} \propto 2^{-Ld(\sigma,\sigma')} = \tau_P^{-d(\sigma,\sigma')}$$

ORIGIN OF UNIVERSALITY

E.g. $N = 3$



transpositions = elementary instantons



$$\sim \frac{(te^{-LI^*})^2}{2!} = \frac{(t/\tau_P)^2}{2!}$$

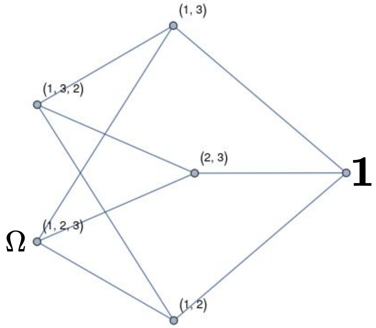
$$\sim te^{-LI^{**}}$$

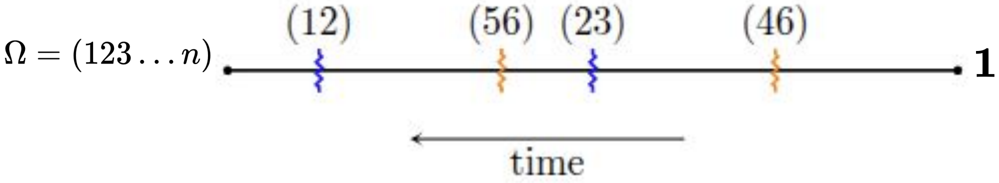
$$(T^t) = (1 + A/\tau_P)^t \sim e^{xA} \ , \qquad A_{\sigma,\sigma'} = \begin{cases} 1 & d(\sigma,\sigma') = 1 \\ 0 & \text{otherwise} \end{cases} \qquad x = t/\tau_P$$

EXAMPLE: Renyi's entropies

$$S_n(t) = \frac{1}{1-n}\overline{\log \mathrm{Tr}[\rho^n(t)]}$$

$$\Omega = (123\ldots n)$$





number of decomposition of a cycle as product of transpositions: n^{n-2}

$$S_n(t) = -\ln\frac{t}{\tau_P} - \frac{1}{1-n}\log\left[\frac{n^{n-2}}{(n-1)!}\right] + \Delta_n(t)$$

$$S_1(t) := -\mathrm{Tr}[\rho(t)\ln\rho(t)] = -\ln\frac{t}{\tau_P} + 1 - \gamma + \Delta_1(t)$$

EXAMPLE: Order by order replica limit

$$\tilde{\rho} = M^\dagger M\,, \qquad M = m_1 m_2 \dots m_T$$

$$\text{IRM}: \quad \overline{S_2} = -\ln x + \frac{4}{3}x^2 - \frac{637}{90}x^4 + O(x^6), \quad N \to 0$$

$$\text{BR}: \quad \overline{S_2} = -\ln x + x^2 - \frac{949}{180}x^4 + O(x^6), \quad N \to 1$$

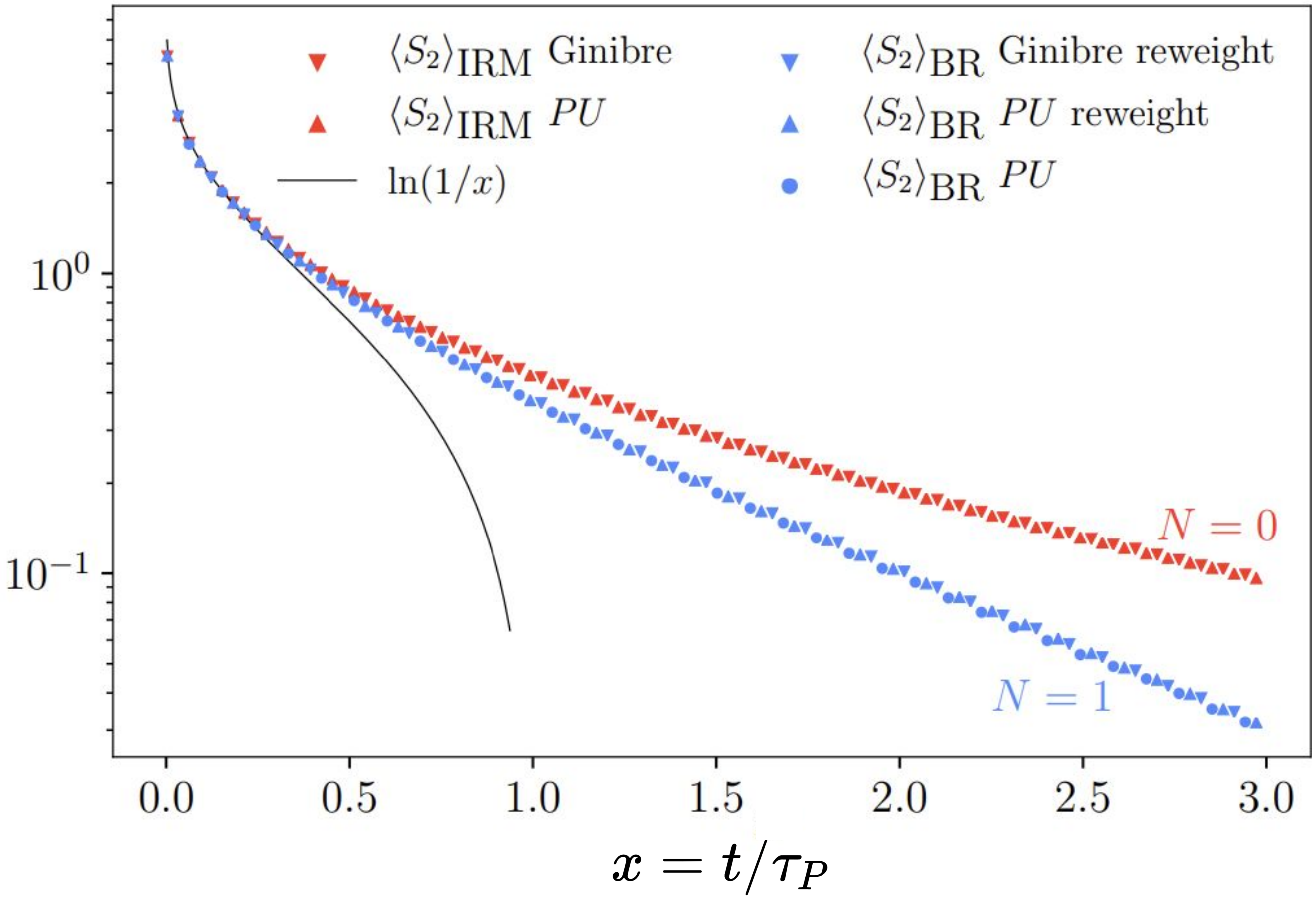
$$\text{IRM}: \quad \overline{S_1} = -\ln x + 1 - \gamma + \frac{11}{24}x^2 - \frac{1739}{2880}x^4 + O(x^6), \quad N \to 0$$

$$\text{BR}: \quad \overline{S_1} = -\ln x + 1 - \gamma + \frac{5}{24}x^2 - \frac{239}{2880}x^4 + O(x^6) \quad N \to 1$$

UNIVERSAL FUNCTIONS DESCRIBING PURIFICATION

$$\mathcal{N} \to \infty, T \to \infty\,, \quad T/\mathcal{N} = T/\tau_P = x$$

NUMERICAL COLLAPSE – S_2



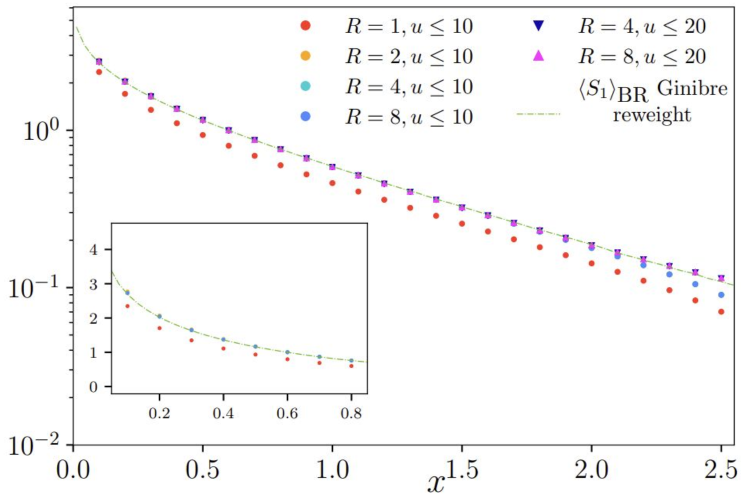
VON NEUMANN ENTROPY

(RESUMMATION IN TERMS OF PLANCHEREL MEASURE)

$$\overline{S_1} = -\overline{\mathrm{tr}[\tilde{\rho}\log\tilde{\rho}]\mathrm{tr}[\tilde{\rho}]}$$

$$\lim_{R\to\infty} G_R(u) = G(u) \qquad G_R(u) = \det(I_{j,k})_{j,k=0}^{R-1},$$

$$\overline{S_1}_{\ \mathrm{BR}} = -\ln\left[2\sinh\frac{x}{2}\right] + 1 - \gamma + \int_0^\infty \mathrm{d}u\,\frac{G(u)-e^{-u}}{u^2},$$



CONCLUSIONS

- MIPT can be seen as **inference for time series – mapping to directed polymer**
- Volume law phase of MIPTs can be studied using RMT
- The long-time dynamics shows universal behavior
- Scaling functions can be computed counting paths in the permutation group
- Universality is relevant in any context involving multiplication of RMs