Eigenstate Correlations, the ETH and Quantum Information Dynamics in Chaotic Many-Body Quantum Systems

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Outline

- Motivation
- Eigenstate Thermalization Hypothesis
- Generic dynamical features in local systems
- Identification of leading order correlators
- Ansatz for JDF of multiple eigenstates
- Numerical results
- Summary and Outlook

Motivation

Thermalization

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Thermalization

Thermalization

- Memory of initial state gets lost
- This works also in a **thermos bottle!**
- Theoretical problem: **unitary dynamics**

Eigenstate Thermalization Hypothesis

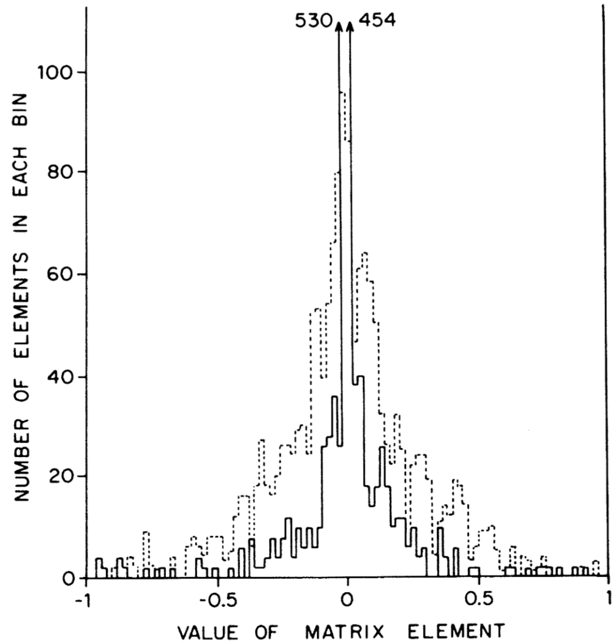
Eigenstate thermalization hypothesis (ETH)

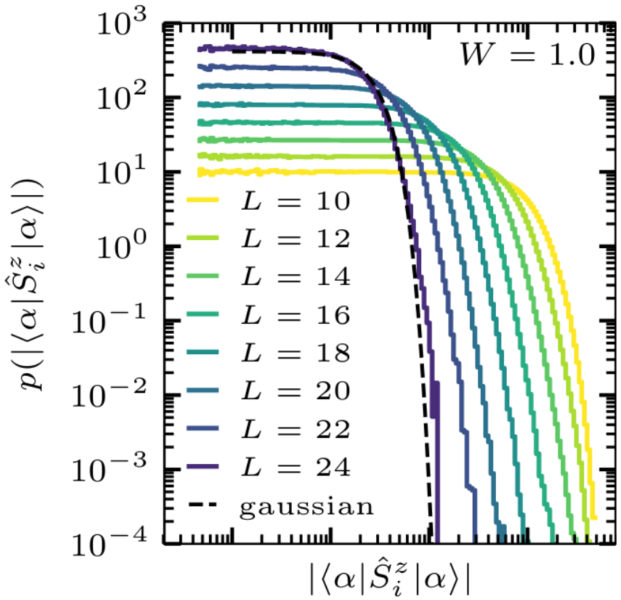
- Initial states $|\psi_0\rangle$ generically evolve into states of the Gibbs ensemble.
- Effective Lagrange multipliers (β, μ etc.) fixed during evolution.
- Eigenstate structure

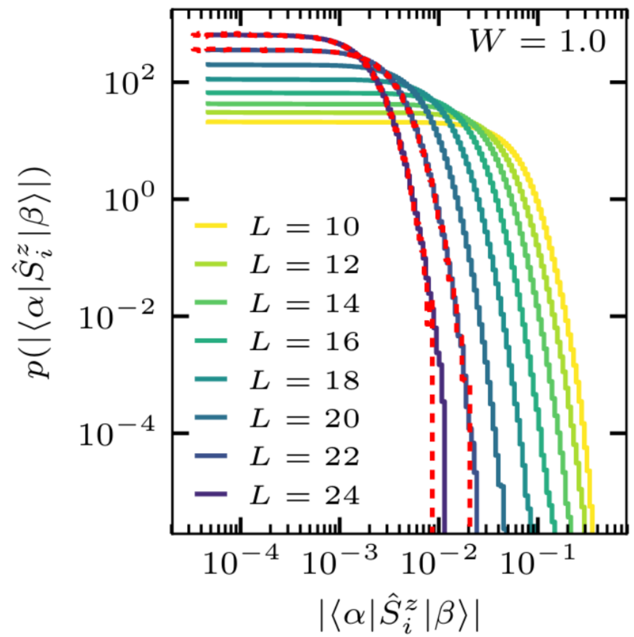
$$\langle n|A|m\rangle = \delta_{n,m}A(E) + \mathrm{e}^{-S(E)/2} f(E,\omega) R_{n,m}$$

- Peres 84, Feingold et al 84, Deutsch 91, Srednicki 94, 98, Rigol et al. 2008
- Reviews:
  - → D'Alessio, Kafri, Polkovnikov, Rigol 2016,
  - → Borgonovi, Izrailev, Santos, Zelevinsky 2016

Gaussian distributions







$$\dim(\mathcal{H}) = 441.$$

anharmonic oscillator

Feingold et al. 1984

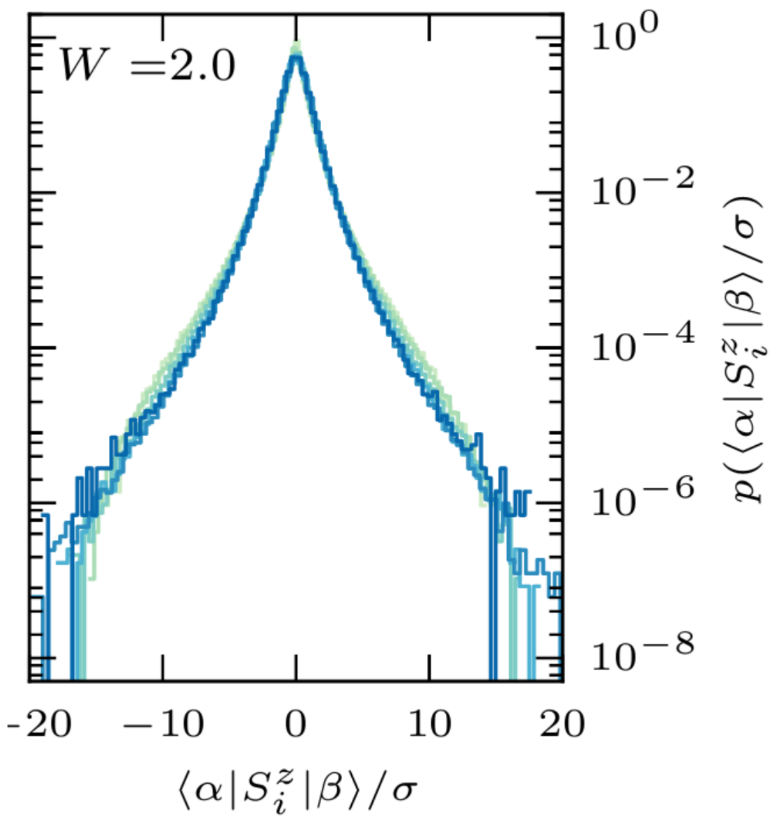
$$\dim(\mathcal{H}) = 2.7 \cdot 10^6.$$
$$h_i \in [-W, W].$$

Luitz, Khaymovich, Bar Lev

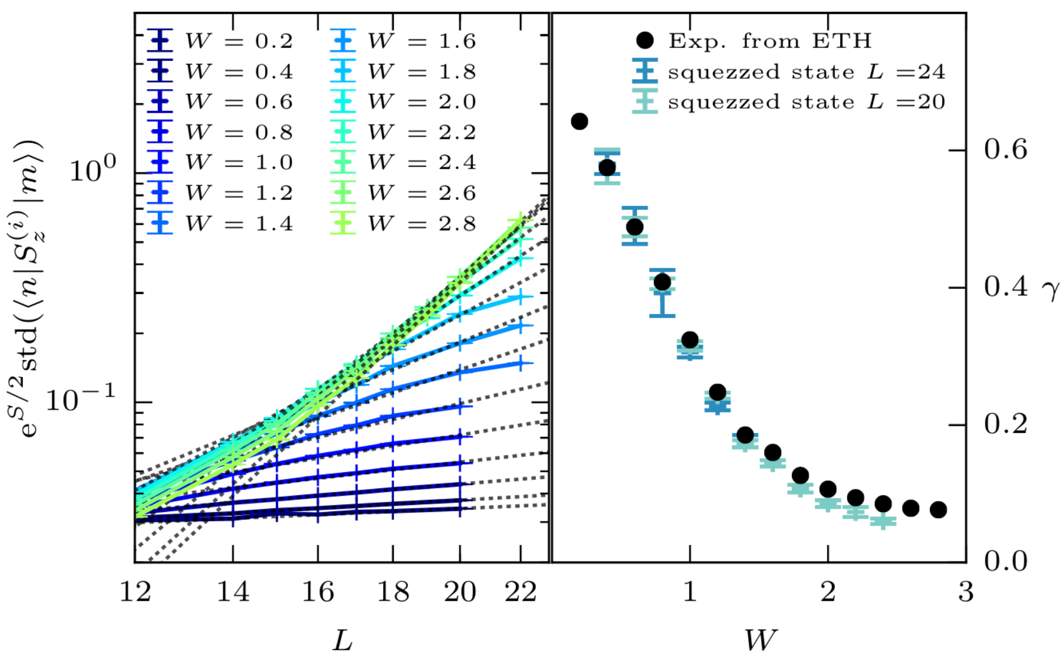
SciPost Phys. Core 2020

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + h_i S_i^z$$

Non-gaussian distributions



- Distributions can be non-gaussian Luitz PRB 2016, Luitz and Bar Lev PRL 2016, Roy et al PRB 2018
- Scaling of variance encodes decay of **autocorrelation function**



Matrix element correlations

Missing element in ETH

- Matrix elements $\langle a|O|b\rangle$ need to be correlated to explain observed Lyapunov exponents in information scrambling Foini and Kurchan, PRE 2019
- Spatial structure in OTOC imply structure beyond ETH Chan, de Luca, Chalker, PRL 2019
- Studied early on Prosen 1993
- Observation of correlations and relation to transport Dymarsky PRL 2022, Richter et al. PRE 2020

Full ETH

$$\overline{O_{12}O_{23}\cdots O_{q1}} = \mathrm{e}^{-(q-1)S(E^+)F^{(q)(\omega)}_{E^+}}, \quad i \neq j$$

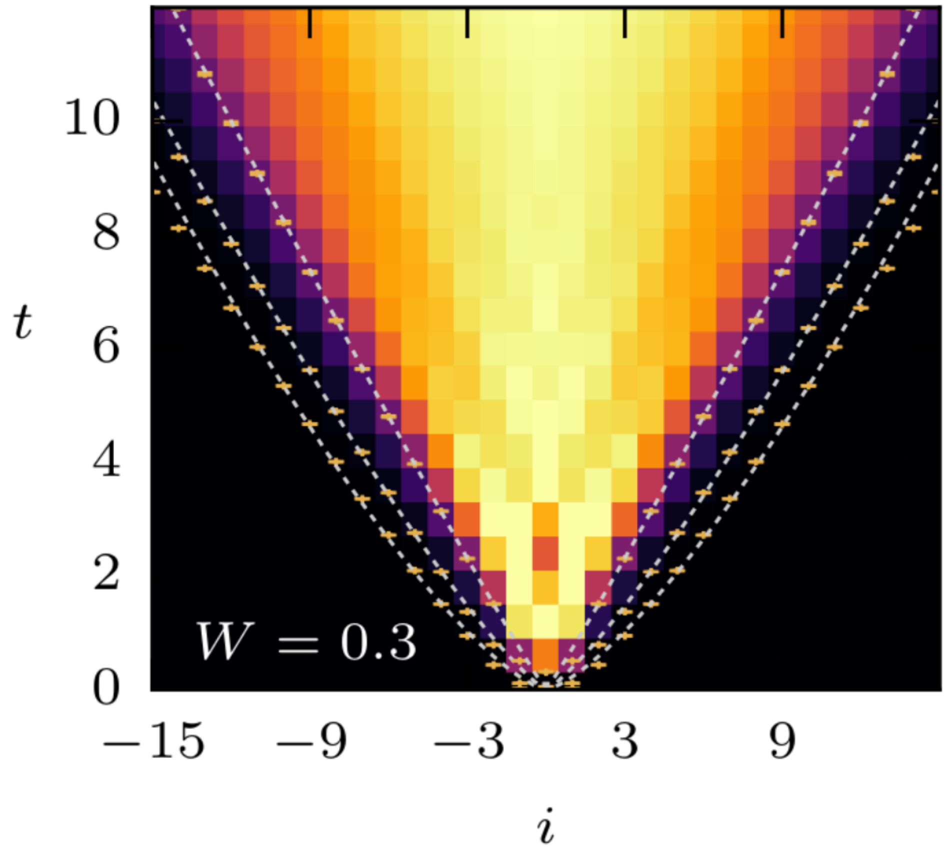
Factorization of average in case of repeated indices Pappalardi, Foini, Kurchan, PRL **129**, 170603 (2022)

Generic dynamical features in local systems

Out of time order corrleators (OTOC)

$$C(t) \propto \|[A_0(t), B_i]\|_F^2$$

- Independent of initial states
- Quantify spreading of **operators**
- Encode fundamental speed limits: Lieb-Robinson bounds Lieb and Robinson 1972



Larkin and Ovchinnikov 1969, Shenker and Stanford 2016, Luitz and Bar Lev 2017

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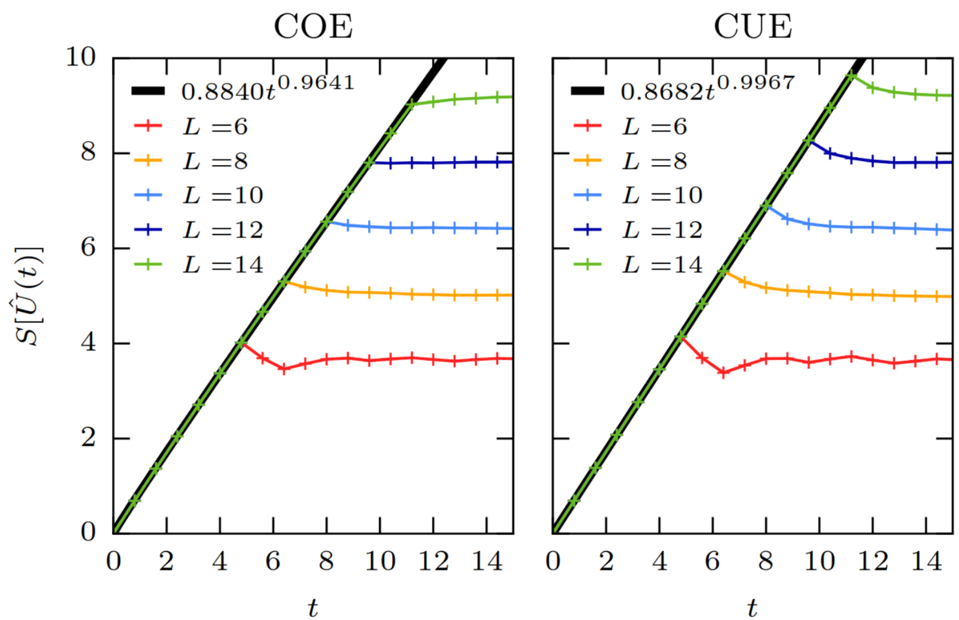
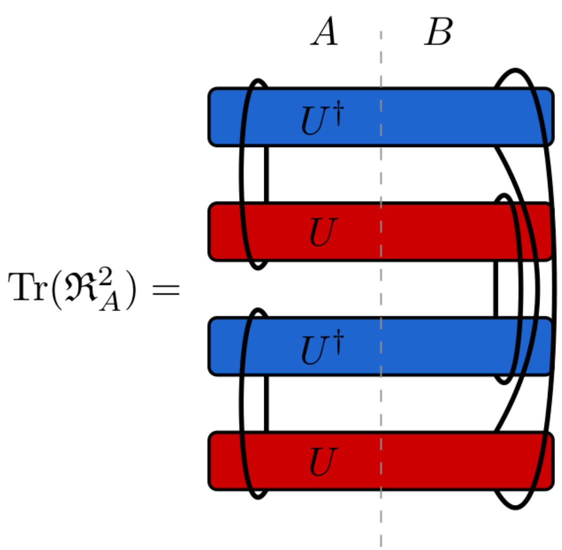
Operator entanglement entropy

independent of choice of operators

- Complexity growth is **property of the time evolution operator** $U(t)$
- Quantify by operator entanglement entropy Prosen and Pižorn 2007, Zhou and Luitz 2017





$$\mathrm{opEE}_\alpha(U) = \frac{1}{1-\alpha}\ln[\mathrm{Tr}(\mathfrak{R}_A^\alpha)].$$

Operator entanglement growth

Hamiltonian case

- Spectrum of time evolution operator $U(t)$ evolves on unit circle
- Eigenvalues move at **different** speeds

$$e^{-iE_m t}$$

- Eigenvectors $|m\rangle$ **do not change**

Operator entanglement growth

Floquet case

- Spectrum of time evolution operator $U(nT)$ evolves on unit circle
- Eigenvalues ω_m evolve as

$$\omega_m^n$$

- Eigenvectors $|m\rangle$ **do not change**
- Phase relation between eigenvectors gets lost over time

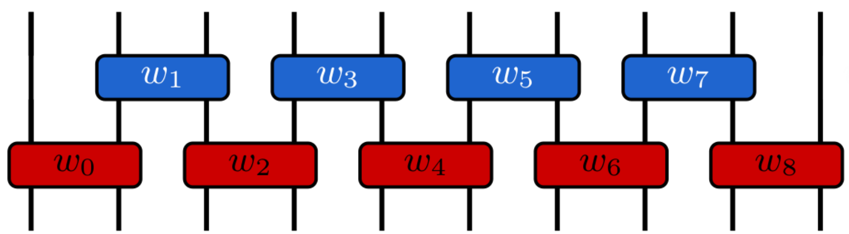
Identification of leading order correlators

Models

Goal: Identify correlators capturing this universal behavior

- Focus on brickwork **Floquet circuits**

$$W(t) = W^t, \quad t \in \mathbb{Z}$$



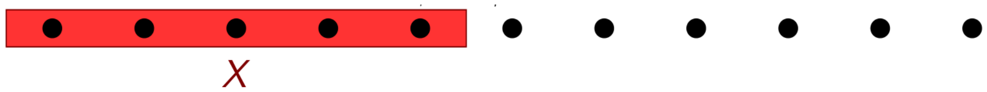
- Spectral decomposition

$$W(t) = \sum_a \mathrm{e}^{-\mathrm{i}\theta_a t} |a\rangle\langle a|.$$

- Absence of energy structure: simplest model
- **No conserved densities**
- Local Hilbert space q
- Gates w_i from truncated Haar ensemble
- Contrast with **global Haar unitary**

Correlators

Simple autocorrelator

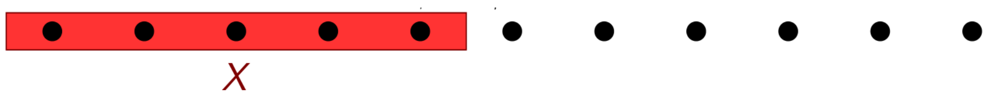
- 

- Introduce operators $X_\alpha, X_\beta, X_\gamma, \ldots$ with **support** $X, Y, \ldots$
- Simplest correlator:

$$q^{-L}\mathbf{Tr}[X_\alpha(t)X_\alpha] = q^{-L}\sum_{ab}|\langle a|X_\alpha|b\rangle|^2 \mathrm{e}^{\mathrm{i}(\theta_a-\theta_b)t}\,.$$

Correlators

OTOC



$$q^{-L}\mathrm{Tr}[X_\alpha(t)Y_\beta X_\alpha(t)Y_\beta] =$$

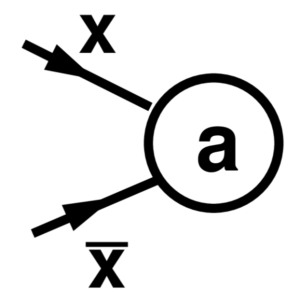
$$q^{-L}\sum_{abcd}\langle a|X_\alpha|b\rangle\langle b|Y_\beta|c\rangle\langle c|X_\alpha|d\rangle\langle d|Y_\beta|a\rangle e^{\mathrm{i}(\theta_a-\theta_b+\theta_c-\theta_d)t}.$$

Depend on choice of X_α, Y_α: **average over all X_α with support on X**

Generic Correlators

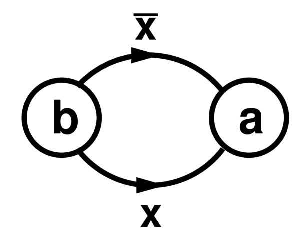
Systematic construction

Basic object: Schmidt matrix $C_X(a)$



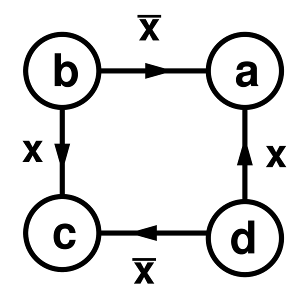
- matrix version of eigenstate $|a\rangle$

$$|a\rangle = \sum_{i_X, i_{\bar{X}}} [C_X(a)]_{i_X, i_{\bar{X}}} |i_X\rangle \otimes |i_{\bar{X}}\rangle.$$



- Lowest order correlator

$$\mathrm{Tr}[C_X(a) C_X^\dagger(b)] = \delta_{ab}$$



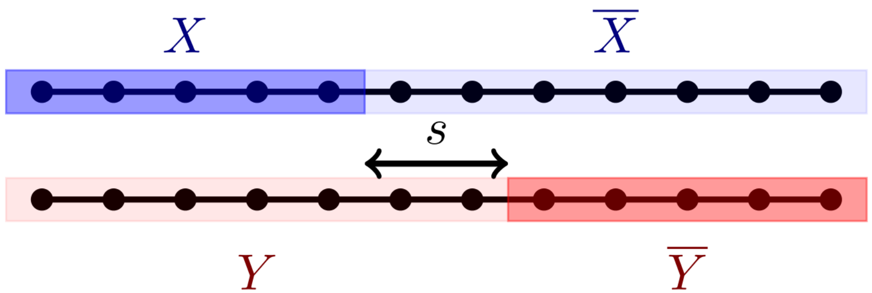
- Next order

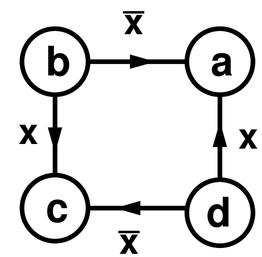
$$M_X(abcd) = \mathrm{Tr}[C_X(a) C_X^\dagger(b) C_X(c) C_X^\dagger(d)].$$

Generic Correlators

Encoding spatial structure

- Use different bipartitions X, Y





- $M_X(abba)$

- phase invar. $|a\rangle \to \mathrm{e}^{\mathrm{i}\phi}|a\rangle$
- Every $C_X(a)$ needs $C_Y(a)^\dagger$.

$$F_2(X, \theta) = q^{-(L+L(X))}\Big[\sum_{ab} M_X(abba)\delta(\theta - \theta_a + \theta_b)\Big]_{\mathrm{av}}$$

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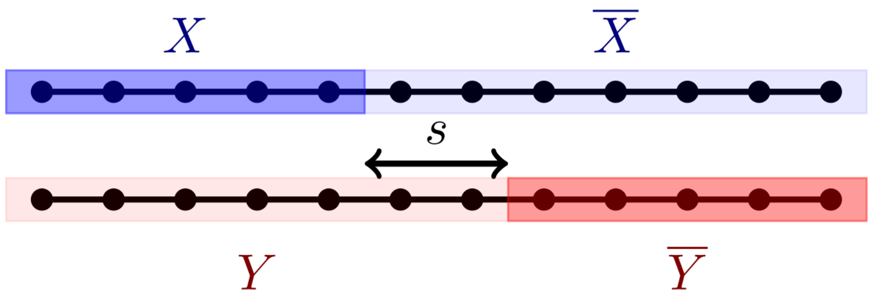
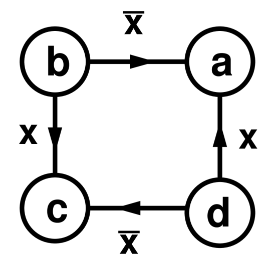
Generic Correlators

Encoding spatial structure

- Use different bipartitions X, Y





- $M_X(abcd)M_Y^*(abcd)$

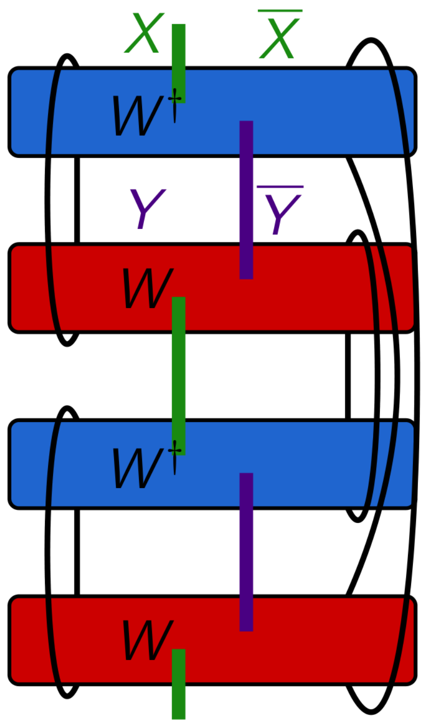
- phase invar. $|a\rangle \to \mathrm{e}^{\mathrm{i}\phi}|a\rangle$
- Every $C_X(a)$ needs $C_Y(a)^\dagger$.

$$F_4(X,Y,\theta) = q^{-L(X,Y)}\Big[\sum_{abcd} M_X(abcd)M_Y^*(abcd)\delta(\theta-\theta_a+\theta_b-\theta_c+\theta_d)\Big]$$

Relation to opEE

Generalize opEE_2 to **different bipartition** for rows and columns of $W(t)$

$\mathrm{Tr}(\mathfrak{R}(X, Y, t)^2) =$



Directly proportional to Fourier transform

$$f_4(X, Y, t) = \int_{-\pi}^{\pi} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\theta t} F_4(X, Y, \theta)$$

Equivalent to OTOC averaged over all operators with support on X, Y

Ansatz for JDF of multiple eigenstates

Describing correlations beyond ETH

- F_2 contains $M_X(abba)$: captures correlations of 2 eigenstates $|a\rangle$, $|b\rangle$
- F_4 contains $M_X(abcd)$ and $M_Y^*(abcd)$: captures correlations of 4 eigenstates $|a\rangle, |b\rangle, |c\rangle, |d\rangle$
- **Joint probability density function (JDF)**

$$P_n(a, b, c, \ldots, n)$$

Joint probability distribution

Haar case

- Random unitary without structure:
  - → Single eigenvector follows isotropic **Haar distribution**

$$P_1^{(0)}(a)$$

  - → n-tuples of **orthogonal** eigenvectors:

$$P_n^0(a, b, c, \ldots, n)$$

Ansatz for JDF

Two eigenvectors

- Locality (and additional structure) **modifies** these JDFs
- Maximum entropy ansatz with **Langrange multipliers** G_n
- **two eigenvectors**

$$P_2(a_1, a_2) = Z_2^{-1} P_2^{(0)}(a_1, a_2) \mathrm{e}^{-S_2(a_1, a_2)}$$

$$S_2(a, b) = \sum_X G_2(X, \theta_a - \theta_b) M_X(abba)$$

- JDF P_2 fixed by **Lagrange multipliers** $G_2(X, \theta)$

Ansatz for JDF

Two eigenvectors

- Locality (and additional structure) **modifies** these JDFs
- Maximum entropy ansatz with **Langrange multipliers** G_n
- **four eigenvectors**

$$P_4(a_1,a_2,a_3,a_4) = Z_4^{-1} P_4^{(0)}(\{a_i\})\mathrm{e}^{-\sum_{j<k} S_2(a_j,a_k) - S_4(\{a_i\})}$$

$$S_4(\{a_i\}) = \sum_{X,Y} G_4(X,Y,\theta_a - \theta_b + \theta_c - \theta_d) M_X(abcd) M_Y(abcd)^*$$

- JDF P_4 fixed by **Lagrange multipliers** $G_4(X,Y,\theta)$

Strategy

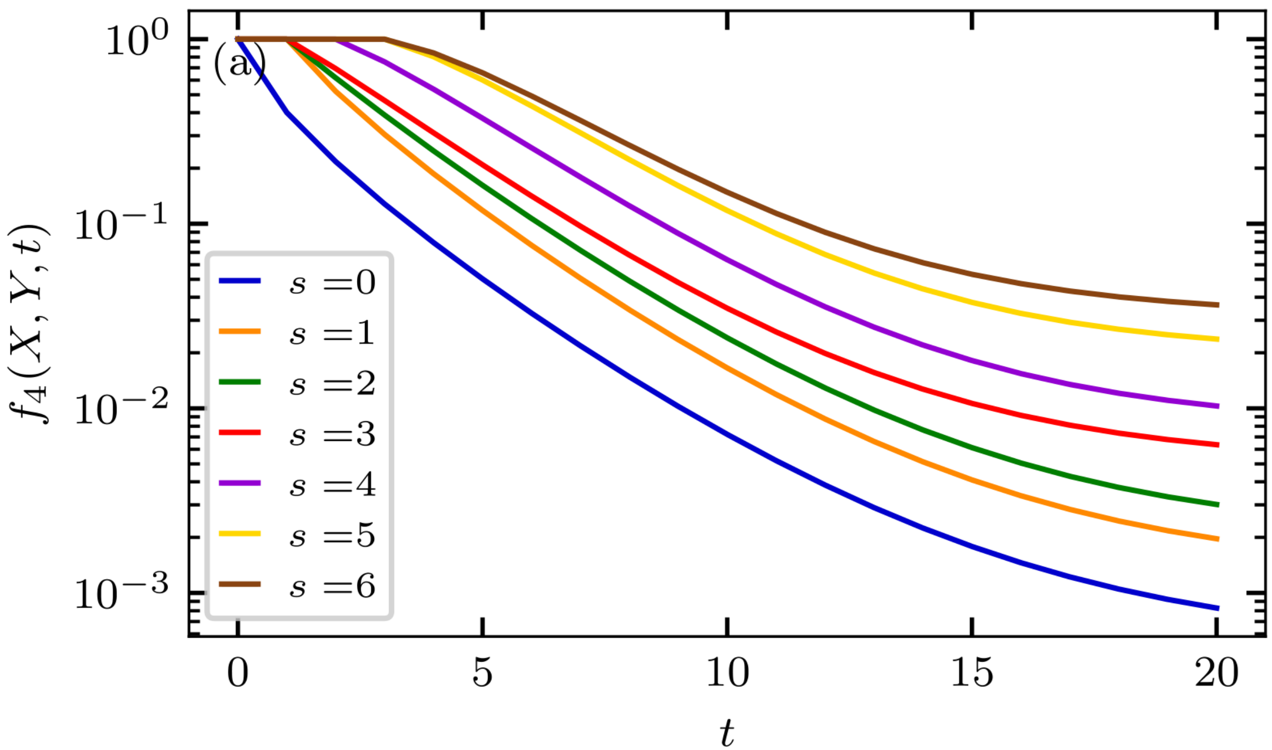
- JDF P_2, P_4 fixed by **Lagrange multipliers** $G_2(X, \theta)$ and $G_4(X, Y, \theta)$
- If JDF is known, $F_4(X, Y, \theta)$ and $F_2(X, \theta)$ can be calculated in principle

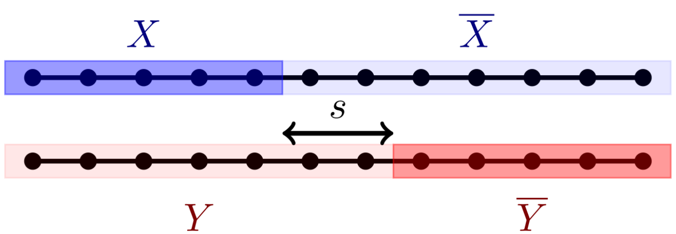
How?

- Random sampling of orthogonal (a, b, c, d) from P_4
- Calculate F_2, F_4 from **exact diagonalization**
- Determine Lagrange multipliers from **comparison**

Numerical results

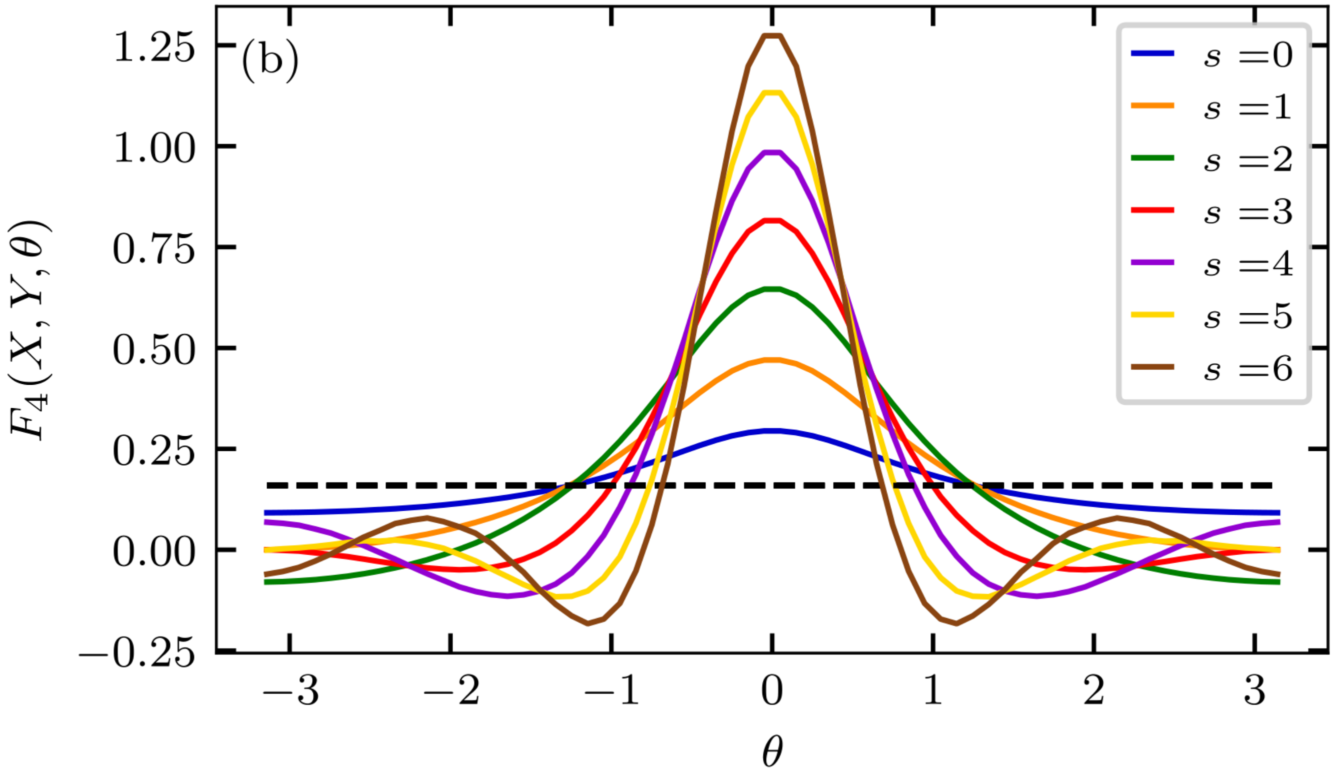
$f_4(X, Y, t)$ from exact diagonalization

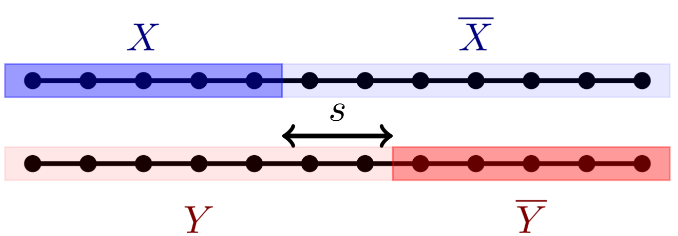




- operator purity decay for $L = 12$, $q = 2$ from ED
- decay timescales depend on subsystem separation s
- encodes spatial structure
- for **Haar unitary:** $\delta(t)$, indep of s

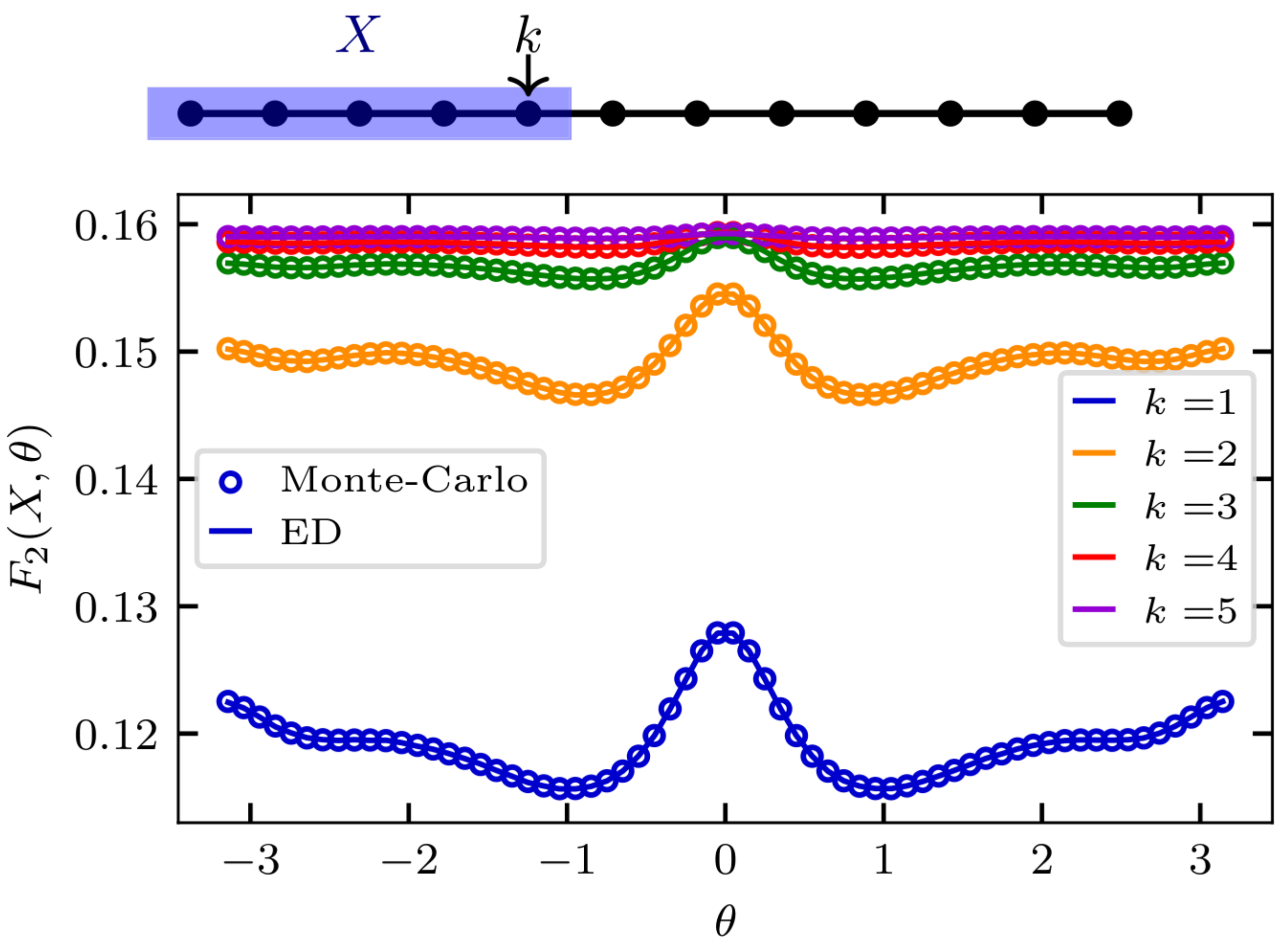
Fourier transform $F_4(X, Y, \theta)$ from ED





- $L = 12, q = 2$ from ED
- $F_4(X, Y, \theta)$, **Fourier transform** of $f_4(X, Y, t)$
- for **Haar unitary**: constant (dashed)

$F_2(X, \theta)$ from exact diagonalization



- $L = 12, q = 2$ from ED
- $F_2(X, \theta)$, has **weak structure** in this system

Determining Lagrange Multipliers

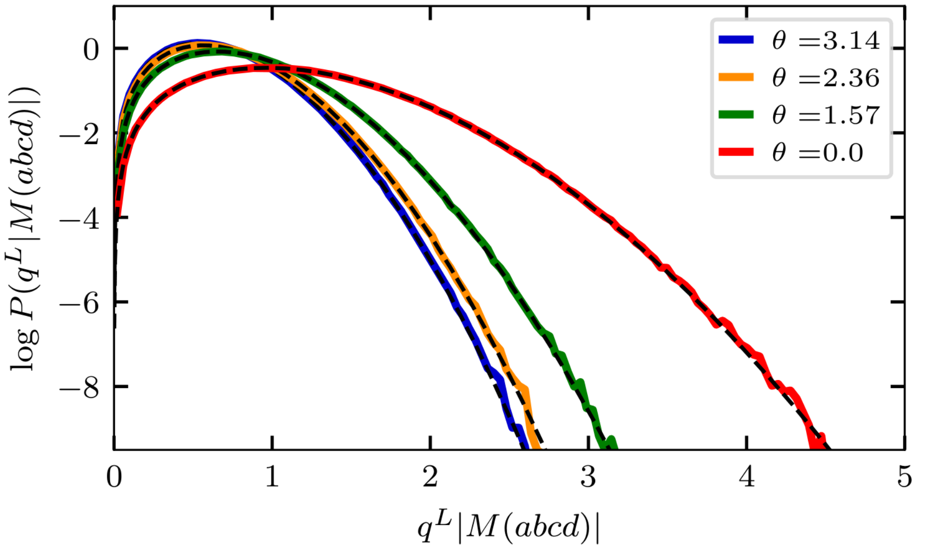
- Negelect weak θ dependence in F_2:

$$G_2(X,\theta) = 0.$$

- Observe: $M_X(abcd)$ **gaussian**
- Organize $M_X(abcd)$ for contiguous, single-cut subsystems in a vector

$$S_4 = M^T G_4 M^*$$

- Relate to covariance of gaussian



Haar:

$$(\mathsf{G}_4^{(0)})^{-1}]_{X,Y} = \left[[\mathsf{M}]_X[\mathsf{M}^*]_Y\right]_0 = q^{L(X,Y}$$

Determining Lagrange Multipliers

For gaussian M, we can show:

$$\left[[\mathsf{M}]_X[\mathsf{M}^*]_Y\right]_{\mathrm{av}} = \left[(\mathsf{G}_4 + \mathsf{G}_4^0)^{-1}\right]_{X,Y}$$

$$[\mathsf{F}_4]_{X,Y} = (2\pi)^{-1} q^{4L-L(X,Y)} \left[[\mathsf{M}]_X[\mathsf{M}^*]_Y\right]_{\mathrm{av}}$$

Allows calculation of G_4 from **exact diagonalization result** for F_4.

Checking if it works

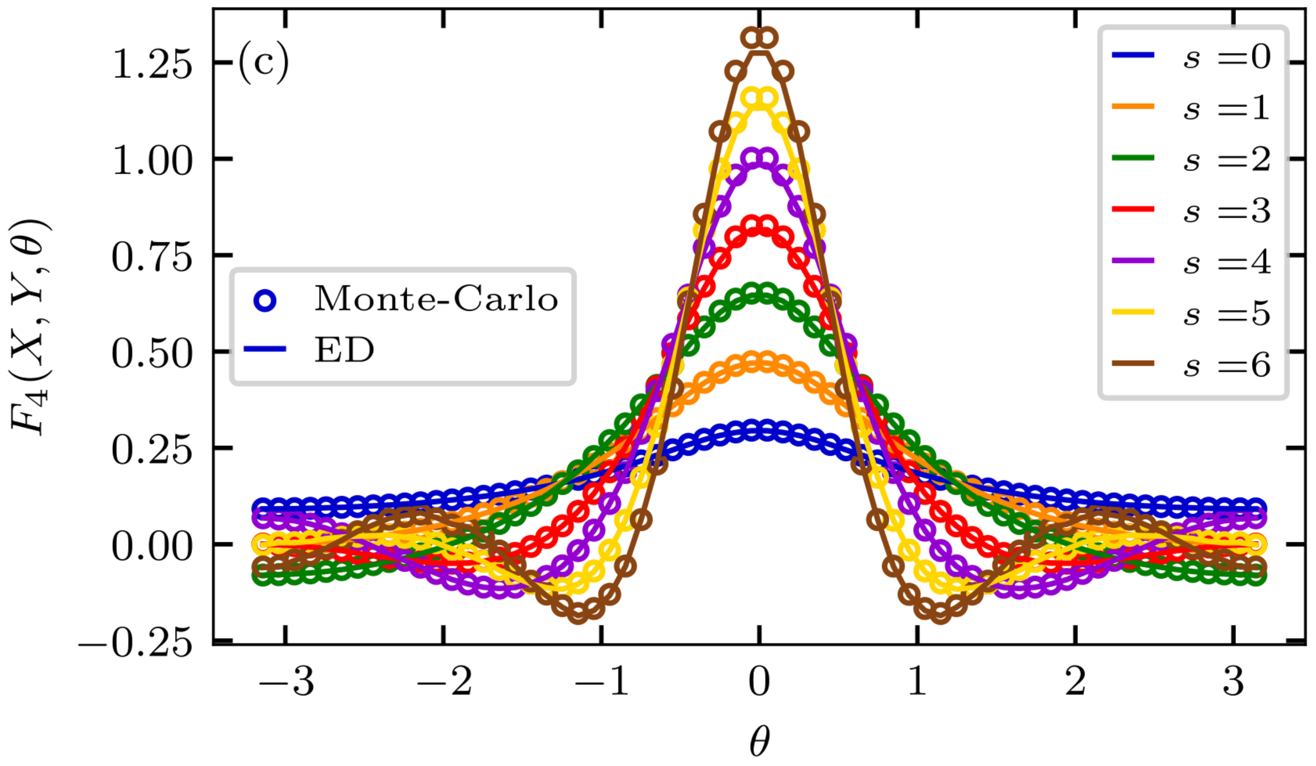
We now have $P_4(a, b, c, d)$, how do we check it?

- Need to calculate $F_4(X, Y, \theta)$ from orthogonal (a, b, c, d) **sampled** according to P_4.

Importance sampling of P_4

- Start with orthonormal, random (a, b, c, d)
- Create new sample by rotation $V = \mathrm{e}^{\mathrm{i}\epsilon A}$: $a' = Va \ldots$
- Accept with $p = \min(1, \mathrm{e}^{-S_4(abcd)+S_4(a'b'c'd')})$ (Metropolis-Rosenbluth)
- $\epsilon = 0.1 \ldots 0.8, A \in \mathrm{GUE}$
- $\approx 10^9$ samples
- Preserves orthogonality of quadruple (a, b, c, d)

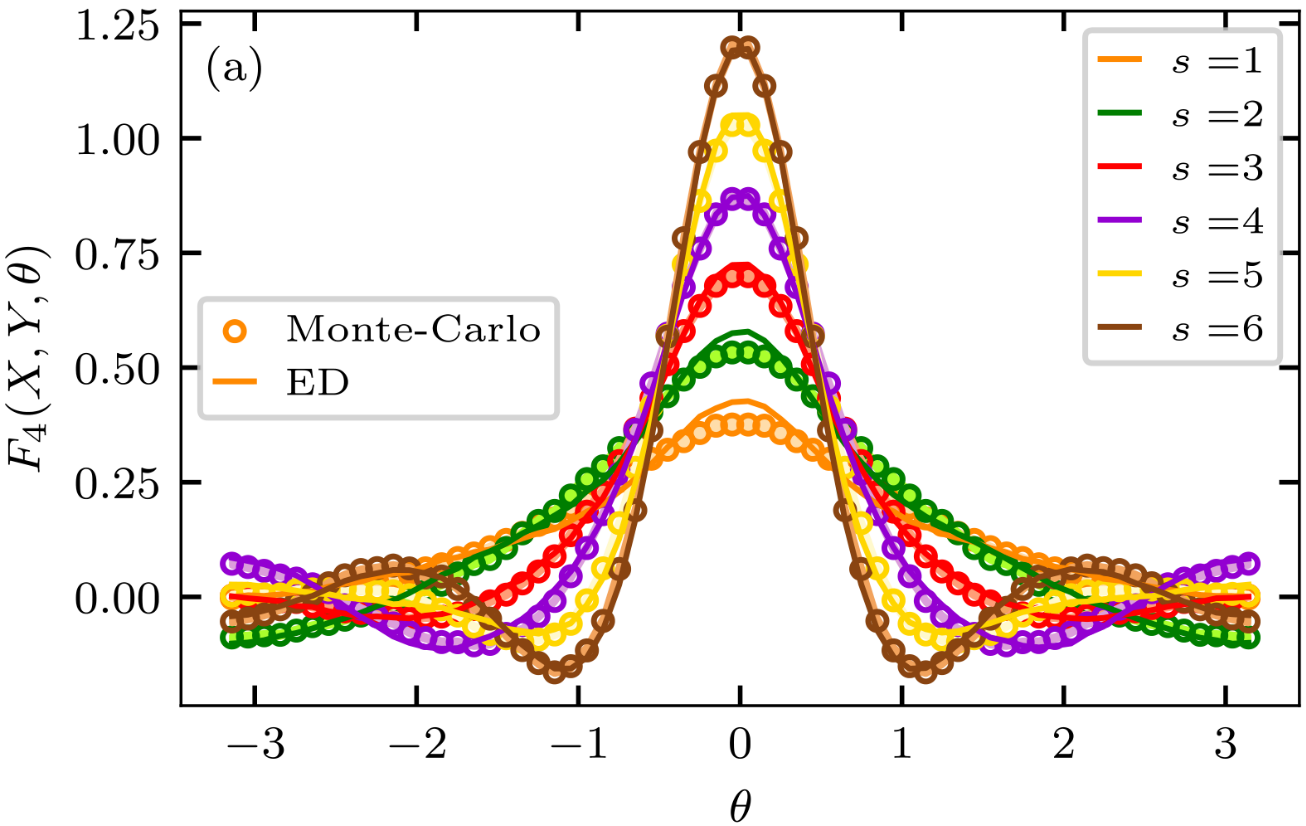
Comparing ED and MC



- $L = 12, q = 2$ ED vs MC
- Very good agreement!
- Larger q generally works better
- Larger separation: boundary effects

Comparing ED and MC

different geometry



- $L = 12, q = 2$ ED vs MC
- **different geometry** not contained in G_4

Summary and Outlook

- Operator spreading in **correlations of 4 eigenpairs**
- Unique correlator encoding spatial structure $[M_X(abcd)M_Y^*(abcd)]_{\mathrm{av}}$
- Extend ETH to include this structure $P_4(abcd) \propto P_4^{(0)}\mathrm{e}^{-M^T G_4 M^*}$
- Numerical check using importance sampling
- Next: Build in conservation laws (add Lagrange multiplier)
- Relation to FETH?

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