Absence of Normal Diffusion in a Disordered Spin Chain

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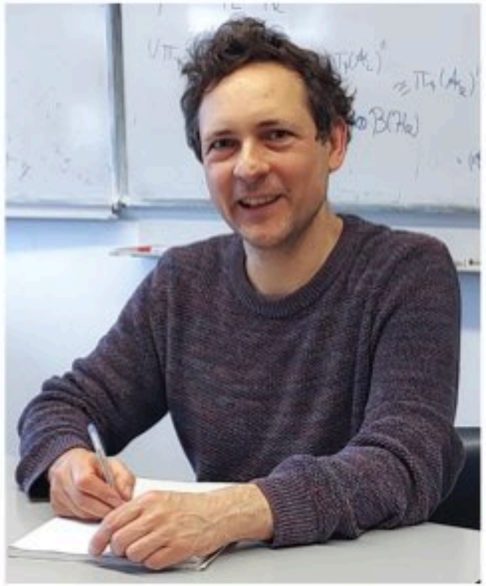
KING'S College LONDON

Inhomogeneous Random Systems, Paris, January 2025

Work in Collaboration with



Lydia Giacomin
Ceremade (Paris)

Oskar Prosniak
Luxemburg

Wojciech De Roeck
KuLeuven (Belgium)

Transport

$$E = \sum_{x=1}^{L} E_x \qquad\qquad \frac{\mathrm{d}E_x}{\mathrm{d}t} = j_{x-1,x} - j_{x,x+1}$$

extensive system conservation law

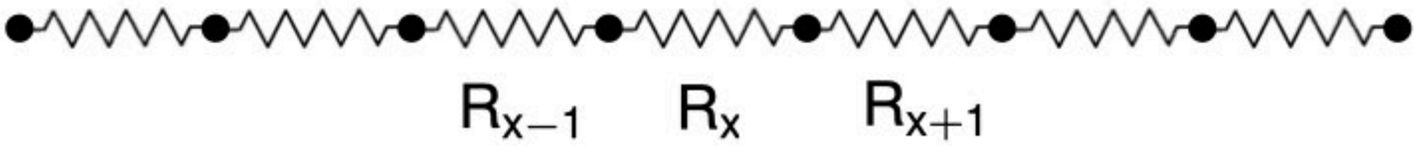
E.g. ***E*** = total energy

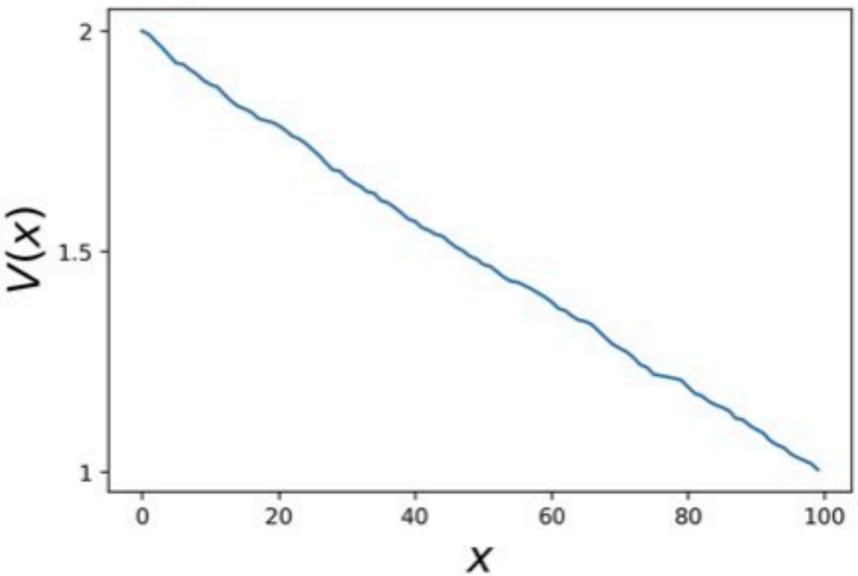
$$J = \langle j_{x,x+1}(t) \rangle_{NESS} \quad \forall t, x$$

Question : How does ***J*** depend on ***L***?

Diffusive Ohm's Law

Resistors in series, i.i.d. (uniform distribution)

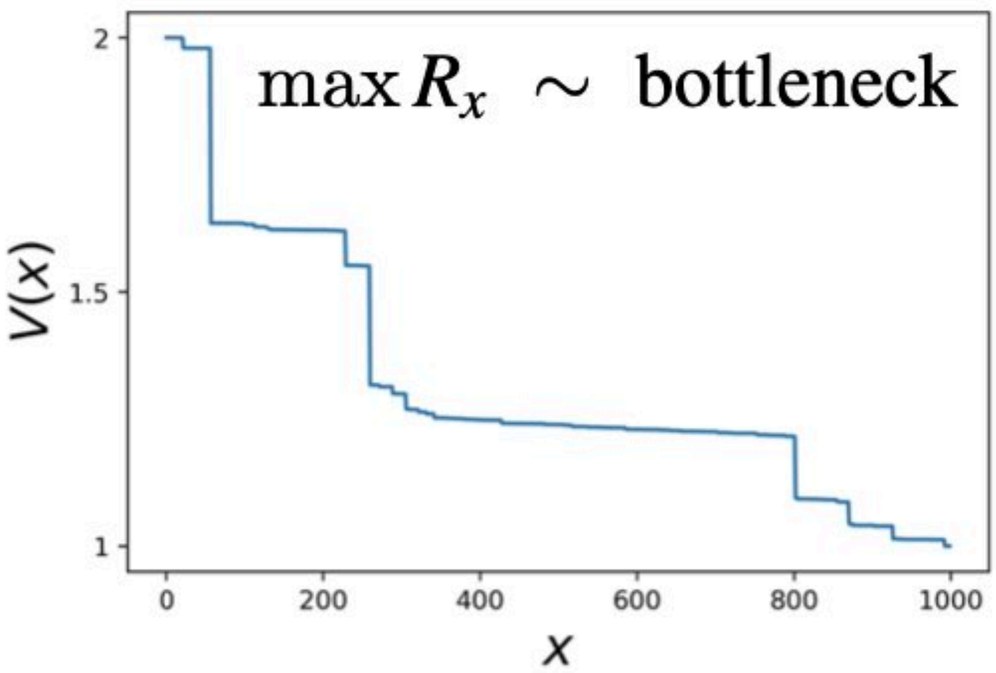




$$J = \frac{\Delta V}{R_1 + \cdots + R_L} \simeq \frac{\Delta V}{L \langle R_1 \rangle}$$

Sub-diffusive Ohm' Law

Resistors, i.i.d. with $\langle R_x \rangle = +\infty$



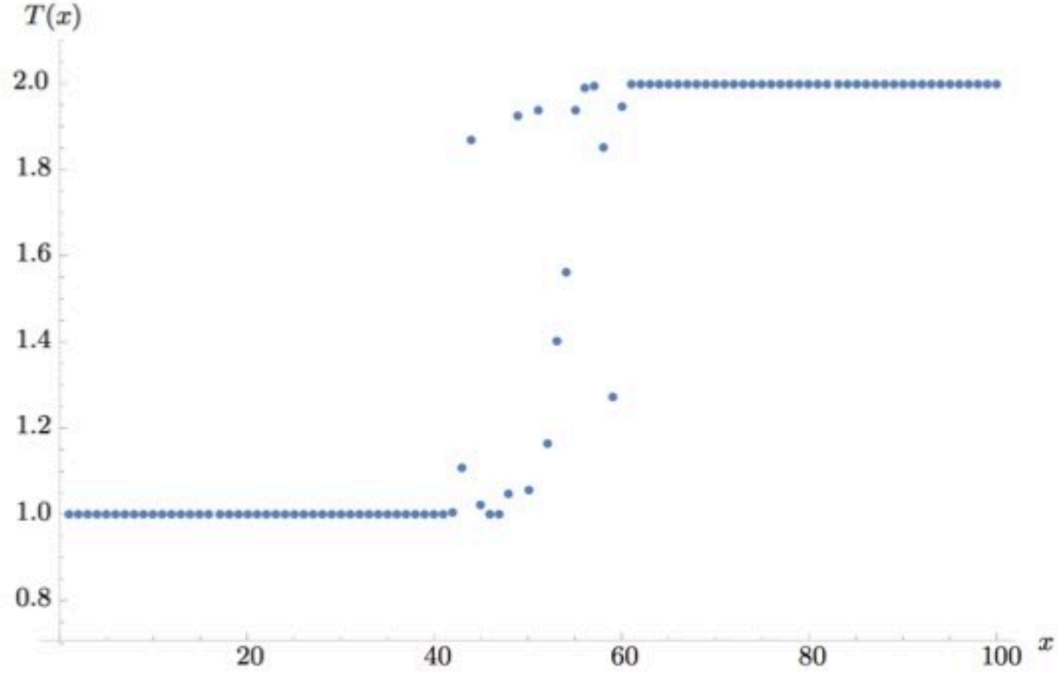
Pareto distribution for R_x

$$J \;=\; \frac{\Delta V}{R_1 + \cdots + R_L} \;\simeq\; \frac{1}{L^{1+a}}\,, \quad a > 0$$

Localization

E.g.: Classical Disordered Harmonic Chain (Free Model):

$$H(p,q) \;=\; \frac{1}{2}\sum_{x=1}^{L}(p_x^2+\omega_x^2 q_x^2) \;+\; g\sum_{x=1}^{L-1}(q_{x+1}-q_x)^2$$

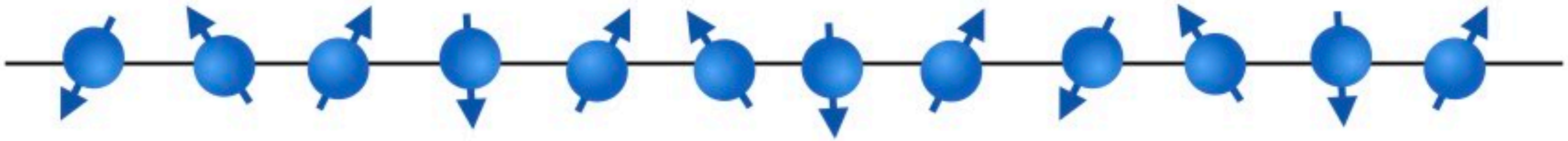


Numerics:
W. De Roeck, A. Dhar, F. Huveneers, M. Schutz, JSP 2017

Mathematical:
Ducatez, ECP 2019

$$J \;\sim\; \mathrm{e}^{-L/\xi}\,\Delta T$$

These 3 Behaviors in a Single Model!

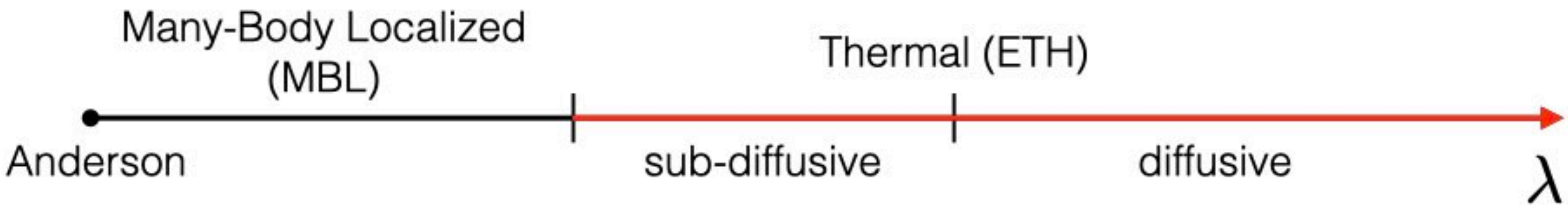


$$H = \sum_{x=1}^{L} h_x Z_x + J \sum_{x=1}^{L-1} Z_x Z_{x+1} + \lambda \sum_{i=1}^{L} X_x$$

classical disordered Ising chain

h_x i.i.d. uniform in $[0,1]$

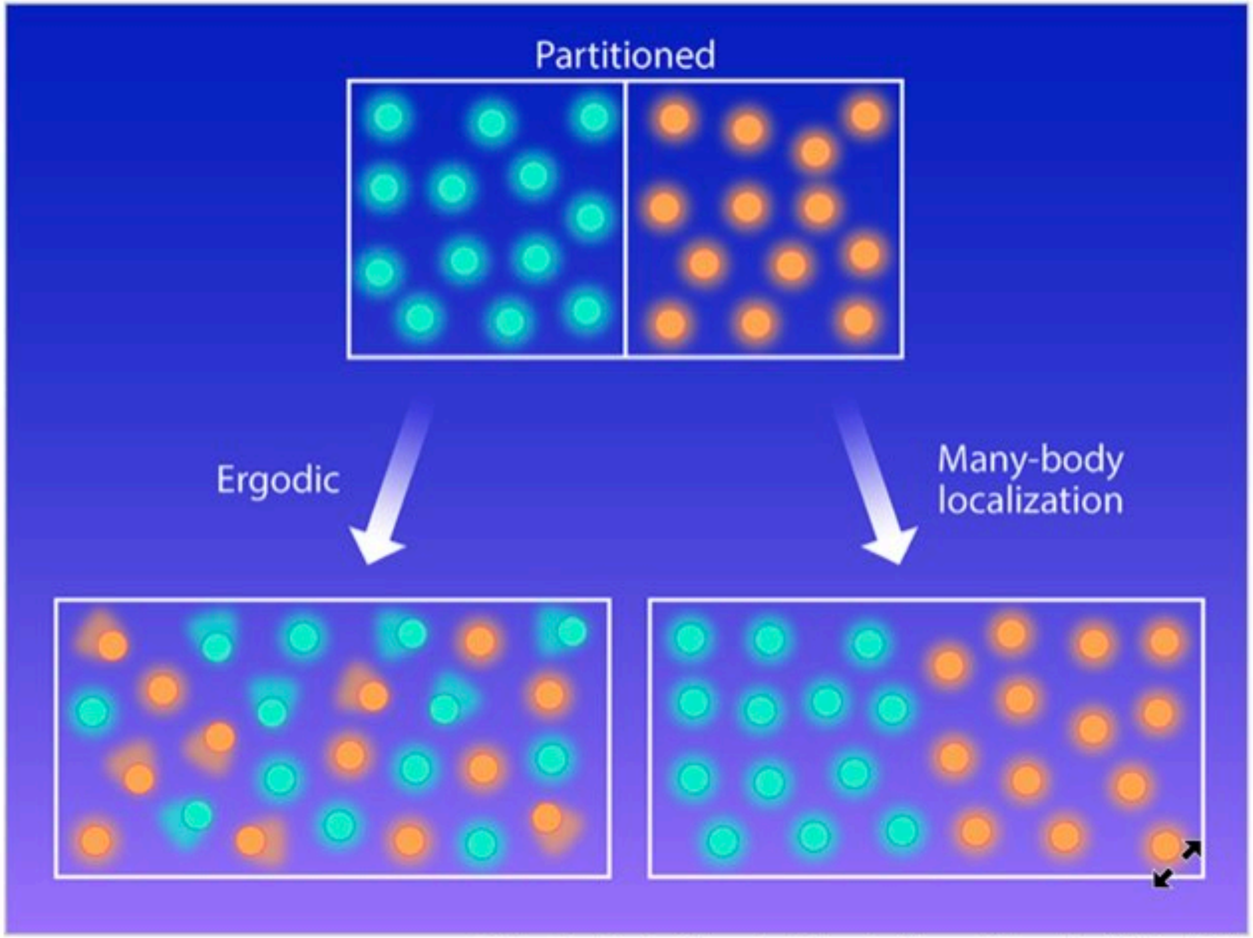
flip-flop



I. Gornyi, A. Mirlin, D. Polyakov, PRL 2005; D. Basko, I. Aleiner, B. Althsuler, AdP 2006; V. Oganesyan, D.A. Huse, PRB 2007, …

MBL / Thermal

MBL/ETH is an **out-of-equilibrium** transition



A. Chandran/Boston University; P. Crowley/Harvard University; APS/Carin Cain

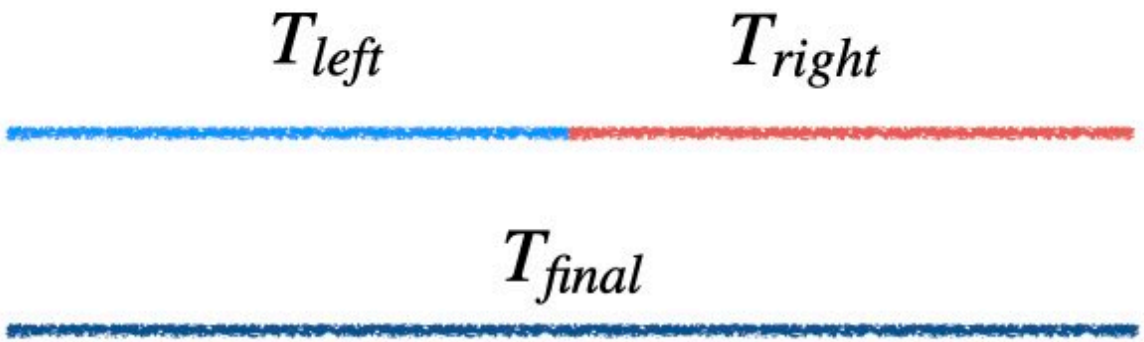
From A. Chandran and P. Crowley (2024), Physics 17

Thermal Phase

Eigenstate thermalization hypothesis (ETH):

$$\langle E|O|E\rangle = \langle O\rangle_T + \mathcal{O}(\mathrm{e}^{-cL}) \qquad (\boldsymbol{O} \text{ local})$$

Physical picture:

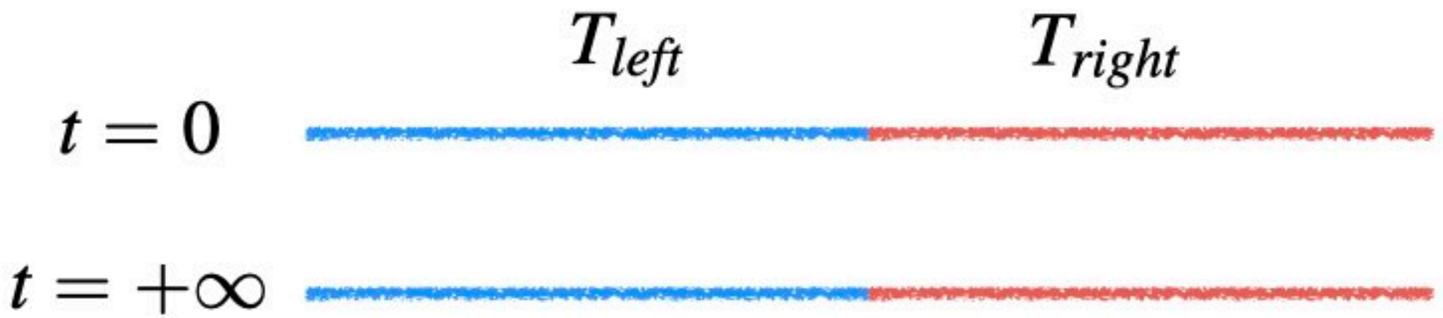


Localized Phase (MBL)

ETH is broken:

$$\langle E|O|E\rangle \neq \langle O\rangle_T \qquad (O \text{ local})$$

Temperatures do not equilibrate, no transport:



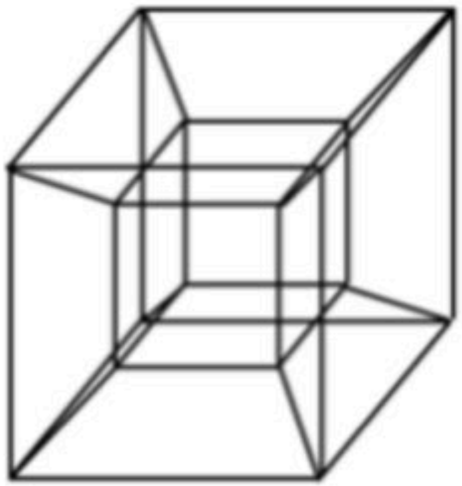
- Closed quantum system that doesn't thermalize on its own: it remembers its initial state forever.
- Robust: "arbitrary" perturbations, emergent integrability.

MBL: Theoretical Description

Anderson point, $\lambda = 0$, eigenstates are product state:

$$|E\rangle \simeq |\uparrow\uparrow\downarrow\uparrow \dots \uparrow\rangle$$

(Too) naive picture for **MBL**, $\lambda > 0$. Localization in Fock space: Eigenstates are small perturbation of product states.



Anderson localization L-dimensional hypercube is **not** quite correct

Local Integrals of Motion

Here is what we can generalize from ***λ=0*** (Anderson point):

- ***L*** integrals of motion: $Z_1, \ldots, Z_L$

$$[Z_x, Z_y] \;=\; 0, \qquad [H, Z_x] \;=\; 0$$

- The Hamiltonian writes as a function of them:

$$H \;=\; \sum_x h_x Z_x + J Z_x Z_{x+1}$$

- They are **local**:

$$[Z_x, O] \;=\; 0 \quad \text{if } x \notin \text{supp}(O)$$

Local Integrals of Motion

Similar picture expected to hold for $0 < \lambda < \lambda_c$:

- *L* integrals of motion: $\mathbf{Z}_1, \ldots, \mathbf{Z}_L$

$$[\mathbf{Z}_x, \mathbf{Z}_y] \;=\; 0, \qquad [H, \mathbf{Z}_x] \;=\; 0$$

- The Hamiltonian writes as a function of them:

$$H \;=\; \sum_{A \subset \{1,\ldots,L\}} J_A \prod_{x \in A} \mathbf{Z}_x$$

- Quasi-**locality**:

$$\|[\mathbf{Z}_x, O]\| \;\leqslant\; C\mathrm{e}^{-r/\xi}, \quad r \;=\; \mathrm{dist}(x, \mathrm{supp}(O))$$

M. Serbyn, Z. Papic, D. Abanin, PRL 2013
D. Huse, R. Nandkishore, V. Oganesyan, PRB 2014

Thermal Phase: Absence of Diffusion

Rare regions, aka **Griffiths regions,** with anomalously large disorder create **bottlenecks** and slow down transport

Thermal material:

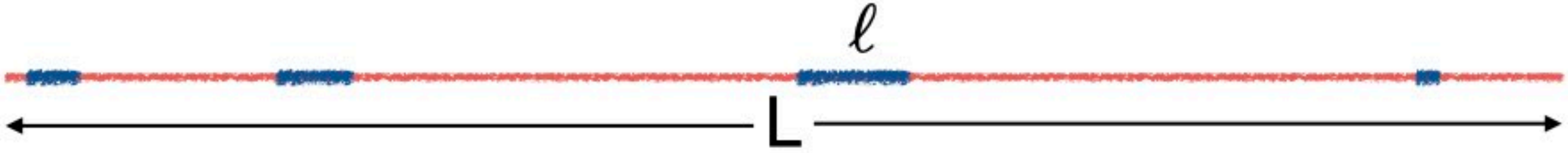
these regions would be localized if isolated

K. Agarwal, S. Gopalakrishnan, M. Knap, M. Müller, E. Demler, PRL 2015,
D. Luitz, N. Laflorencie, F. Alet, PRB 2016

Thermal Phase: Absence of Diffusion

L : total length

ℓ : length of the biggest resistance



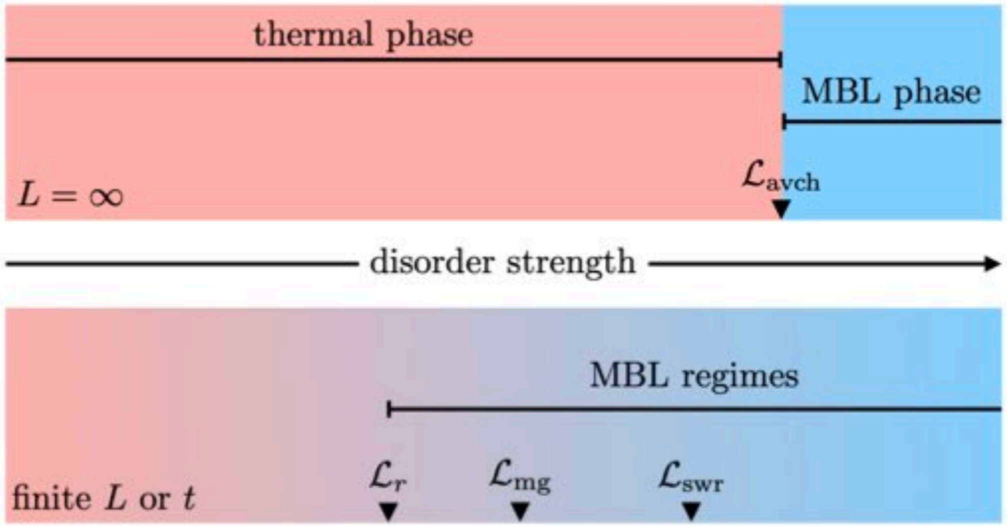
$$\ell = K \log L$$

$$J \sim \mathrm{e}^{-\ell/\xi} = \mathrm{e}^{-(K/\xi)\log L} = \frac{1}{L^{K/\xi}}$$

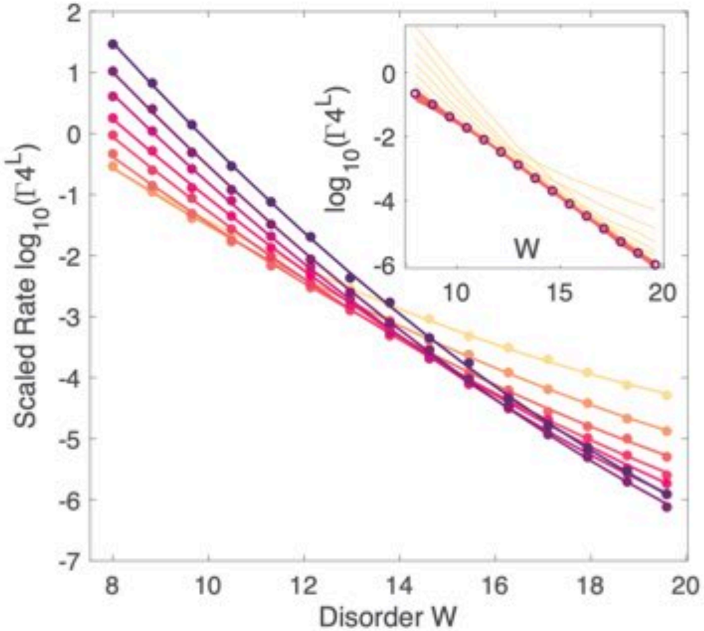
$K/\xi > 1$ near the transition

Need for Math: MBL is debated

Avalanches: The MBL/ETH transition point could be located at much higher disorder values than initially thought.



From A. Morningstar, L. Colmenarez, V. Khemani, D.J. Luitz, D.A. Huse, PRB 2022



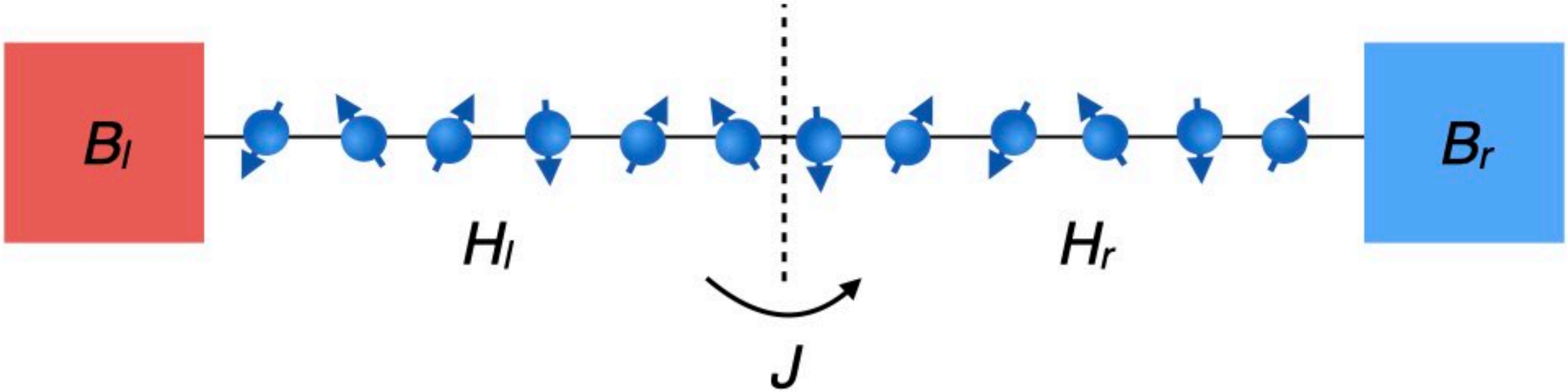
The crossing point drifts substantially from around $W^* \approx 8$ for the smallest system to $W^* > 20$ for the largest available system size.

From D. Sels, PRB 2022

See also B. Krajewski, L. Vidmar, J. Bonca, M. Mierzejewski, PRL 2022

Theorem: Absence of Diffusion

Couple the chain with baths at the boundaries:



$$H_{\mathrm{tot}} = H_{B,\mathrm{l}} + V_{B,\mathrm{l}} \otimes X_1 + H_{B,\mathrm{r}} + V_{B,\mathrm{r}} \otimes X_L + H_L$$

$$J = i[H_{\mathrm{tot}}, H_{\mathrm{l}}] = i[H_{\mathrm{r}}, H_{\mathrm{l}}]$$

Theorem: Absence of Diffusion

Long time average of the current for arbitrary large baths:

$$\langle J_L \rangle := \limsup_{T\to\infty} \sup_B \sup_\rho \frac{1}{T}\int_0^T dt\, \mathrm{Tr}(\rho J(t))$$

Long time average

Supremum over all baths

Theorem (W. De Roeck, L. Giacomin, F. H., O. Prosniak)
If $\lambda > 0$ is small enough,

$$\lim_{L\to\infty} \mathbf{E}\left(L\langle J_L\rangle\right) = 0$$

Remarks: 1. In particular, there is no diffusion in the NESS, if the NESS sets in.
2. Valid for a whole class of Hamiltonians (robustness).
3. First claimed by Gornyi, Mirlin, Polyakov ('05), Basko, Aleinner, Altshuler ('06)

The Mathematics of MBL

- Mathematical Approach pioneered by J. Imbrie (2016)
- Diagonalization needs to preserve locality:

$$H = U^\dagger D U$$

If O is a local operator, then

$$U^\dagger O U = \sum_I O_I, \qquad \|O_I\| \leqslant C e^{-|I|/\xi} \|O\|.$$

- This goes through a renormalization procedure, cf. KAM, Schrieffer-Wolff:

$$U = \lim_{k\to\infty} e^{A^{(k)}} \dots e^{A^{(1)}}$$

Constructing the First Rotation

Define $A^{(1)} = \lambda A$ to diagonalize $H = E + \lambda V$ in the 1^{st} order in λ:

$$
\begin{aligned}
e^{-\lambda A} H e^{\lambda A} &= e^{[\lambda A, \cdot]} H \\
&= H + \lambda [A, H] + \mathcal{O}(\lambda^2) \\
&= E + \underbrace{\lambda V + \lambda [A, E]}_{\text{Cancel this!}} + \mathcal{O}(\lambda^2)
\end{aligned}
$$

We can now define ***A***:

$$
V = \sum_x X_x, \qquad A = \sum_x X_x \frac{1}{\Delta E_x}
$$

Energy denominators

$$
\Delta E_x = 2 Z_x (h_x + J Z_{x-1} + J Z_{x+1})
$$

The Role of the Disorder

- For typical values of the disorder, denominators are large:

$$\Delta E_x = 2Z_x(h_x + JZ_{x-1} + JZ_{x+1})$$

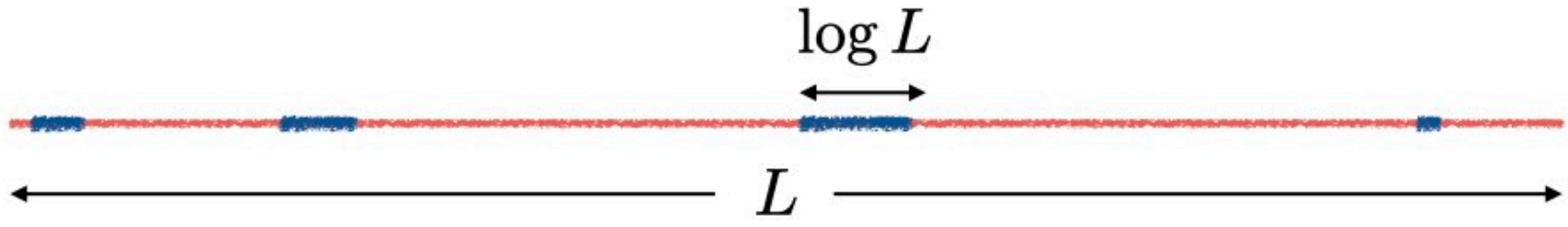
- They become nearly 0 for atypical values: **Resonances**
- Controlling resonances is a challenging aspect:

  Hilbert space dimension: 2^L
  Number of independent variables: L

- Non-perturbative control needed to prove MBL: cf. the **Limited Level Attraction** Hypothesis used by J. Imbrie.
- Proving absence of normal diffusion requires less…

MBL Away from Resonances

Absence of diffusion requires MBL on *atypical* stretches:



Theorem (W. De Roeck, L. Giacomin, F. H., O. Prosniak):

On a stretch of length ℓ, there is MBL with probability

$$P \geqslant e^{-\lambda^c \ell}$$

Thus, we identify a non-resonant set in disorder space and prove MBL on this set

Convergence of the Expansion

First step: $V^{(0)} = \sum_x X_x, \quad A^{(1)} = \sum_x A_x^{(1)}$

$$H^{(0)} = E + \lambda V^{(0)}$$

$$A_x^{(1)} = X_x \frac{1}{\Delta E_x},$$

$$H^{(1)} = \mathrm{e}^{\lambda[A^{(1)},\cdot]} H^{(0)} \simeq E + \lambda^2 [A^{(1)}, V^{(0)}]$$

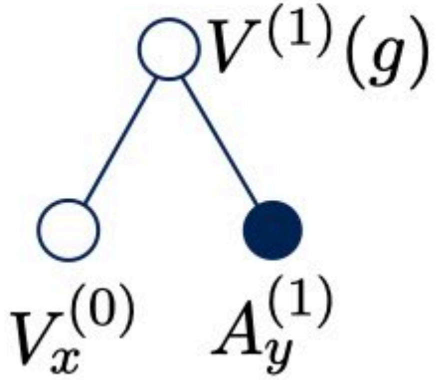
$$=: E + \lambda^2 V^{(1)}$$

The scheme is naively **quadratic**

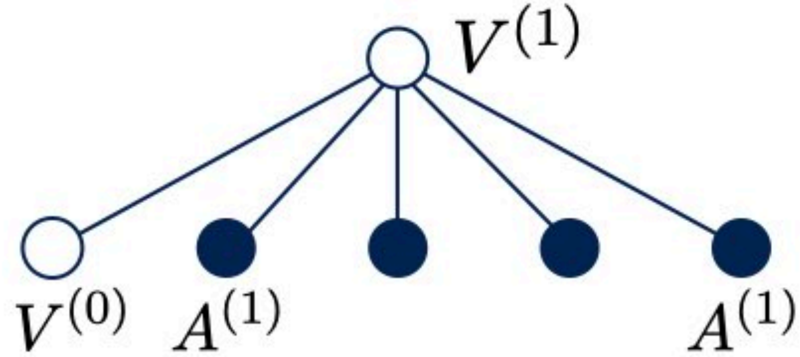
Convergence of the Expansion

Diagrammatic expansion for $\boldsymbol{V^{(1)}}$:

$$V^{(1)} = \sum_g V^{(1)}(g)$$



Only such diagrams appear within our approximation



Other diagrams are generated in general

Convergence of the Expansion

At step ***k***: $V^{(k)} = \sum_g V^{(k)}(g), \quad A^{(k+1)} = \sum_g A^{(k+1)}(g)$

$$H^{(k)} = E + \lambda_k V^{(k)}$$

$$A^{(k+1)}(g) = V^{(k)}(g) \frac{1}{\Delta E_g}$$

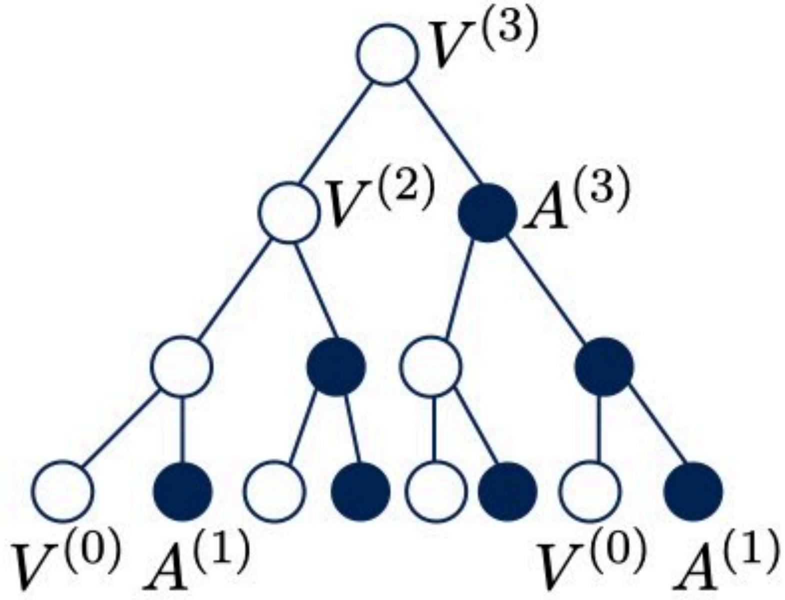
$$H^{(k+1)} \simeq E + \lambda_k^2 [A^{(1)}, V^{(0)}] =: E + \lambda_{k+1} V^{(k+1)}$$

The **naive** flow of the coupling constant is **quadratic**:

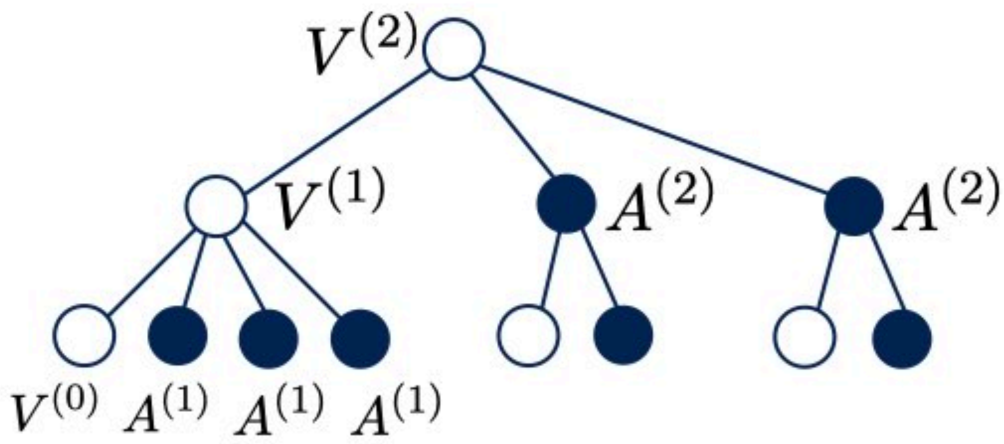
$$\lambda_{k+1} = \lambda_k^2 = \ldots = \lambda^{2^k}$$

Convergence of the Expansion

We can construct a diagrammatic expansion:



Only dyadic diagrams are generated in the approximation above



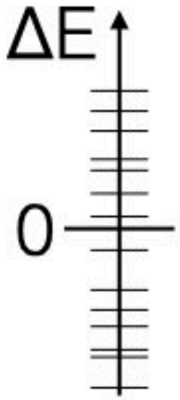
In general, other types of diagrams can be generated

Convergence of the Expansion

We still have to impose non-resonance condition:

At step ***k***, terms may involve up to $\ell_k = 2^k$ spins

Resonances involve 2^{ℓ_k} configurations of these spins:



ΔE can typically get

as small as $2^{-\ell_k} = 2^{-2^k}$

We set a resonance threshold:

$$\Delta E \geqslant \varepsilon_k, \qquad \varepsilon_k := \varepsilon^{\ell_k}$$

Convergence of the Expansion

Flow of the effective coupling constant:

$$\overline{\lambda}_{k+1} \;=\; \overline{\lambda}_k \frac{\overline{\lambda}_k}{\varepsilon_k}$$

This is solved explicitly:

$$\log \overline{\lambda}_{k+1} \;=\; -2^{k+1} \log \frac{1}{\lambda} + (k+1) 2^k \log \frac{1}{\varepsilon}$$

Even for $\boldsymbol{\lambda} \ll \boldsymbol{\varepsilon}$, the second term dominates for large ***k***:
The expansion **seems** only **asymptotic**!

Convergence is non-trivial even away from resonances

Fixing the Convergence

Following a strategy already present in Imbrie's work:

1. **Crowded** diagrams: They allow for a larger resonance threshold and can be estimated **inductively**.

2. **Non-crowded** diagrams: There is (almost) one disorder variable for each denominator, and they can be estimated through a non-inductive, **probabilistic** bound:

$$\mathbb{P}\left(A \geqslant \left(\frac{1}{\varepsilon}\right)^{2^n}\right) \leqslant \varepsilon^{\alpha 2^n}\mathbb{E}(\left(\frac{1}{|\Delta E_1 \ldots \Delta E_{2^n}|^{\alpha}}\right)$$
$$\simeq (C\varepsilon)^{\alpha 2^n}$$

Markov bound with fractional moment: 0<α<1

Conclusion

- We are developing the mathematical study of MBL.
- We establish the **absence of diffusion** at the mathematical level of rigor for generic 1d disordered quantum spin chains.
- Our work takes inspiration from the approach pioneered by J. Imbrie.
- Our work provides a way to interpret some numerical studies that questioned the existence of MBL.