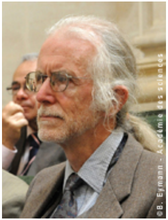
Celebrating the Abel prize of Michel Talagrand

Workshop on inhomogeneous random systems

Institut Henri Poincaré



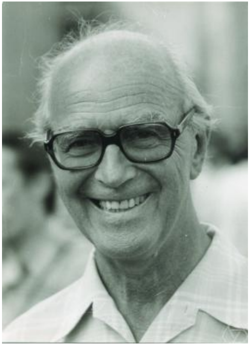
Michel Talagrand

Michel Talagrand
Abel prize 2024

"for his groundbreaking contributions to probability theory and functional analysis, with outstanding applications in mathematical physics and statistics."

Early years

Born in 1952.
Grows up and does undergrad (incl. "agrégation") in Lyon.
Enters CNRS in 1974.



PhD thesis under Gustave Choquet, defended in 1977, in measure theory, topology and geometry of Banach spaces.

"*A problem-solving machine*" (G. Choquet).

Uniform laws of large numbers

$$\sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{k=1}^{n} f(X_k) - \mathbb{E}[f(X_1)] \right| \to 0 \qquad ?$$

Suprema of Gaussian processes

Gilles Pisier joins the group in 1983 and suggests Talagrand to study suprema of Gaussian processes.

$(X_t)_{t\in T}$ a centered Gaussian process. When do we have

$$\mathbb{E}\sup_{t\in T} X_t < +\infty \quad ?$$

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View T as an abstract metric space via

$$d(s,t) := \mathbb{E}[(X_t - X_s)^2]^{\frac{1}{2}}.$$

Andrey Kolmogorov, Richard Dudley (entropy-number upper bound), Volodya Sudakov (lower bound)...

Dudley's upper bound

Let $\mathcal{N}(T, d, \varepsilon)$ be the minimal number of balls of radius ε needed to cover T.

Theorem (Richard Dudley)

There exists an absolute constant $C < +\infty$ such that

$$\mathbb{E} \sup_{t \in T} X_t \leqslant C \int_0^{+\infty} \sqrt{\log \mathcal{N}(T, d, \varepsilon)} \, \mathrm{d}\varepsilon.$$

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Suppose T finite, and for each $n \in \mathbb{Z}$, let T_n be a 2^{-n}-net of T of minimal size. Denote by $\pi_n(t)$ a point of T_n that is closest to $t \in T$. We have

$$X(t) - X(t_0) = \sum_{n=-\infty}^{+\infty} \big(X(\pi_n(t)) - X(\pi_{n-1}(t)) \big),$$

and

$$\mathbb{E} \sup_{t \in T} \big(X(\pi_n(t)) - X(\pi_{n-1}(t)) \big) \leqslant C 2^{-n} \sqrt{\log \mathcal{N}(T, d, 2^{-n})}.$$

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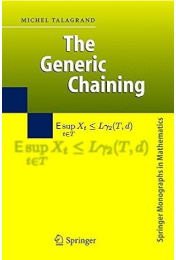

Talagrand shows matching lower bound (1987).
Later: "Majorizing measures without measures" (2001): emphasis on the generic chaining.

Suprema of Gaussian processes

$$\gamma_2(T,d) := \inf \sup_{t\in T} \sum_{n=0}^{+\infty} 2^{n/2} d(t, T_n),$$

where inf is over all $(T_n)_{n\geqslant 0}$, $T_n \nearrow T$ such that $|T_n| \leqslant 2^{2^n}$.

Theorem (Michel Talagrand)

$$C^{-1}\gamma_2(T,d) \leqslant \mathbb{E}\sup_{t\in T} X_t \leqslant C\gamma_2(T,d).$$

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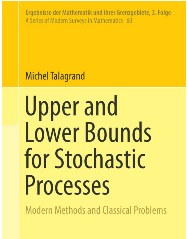
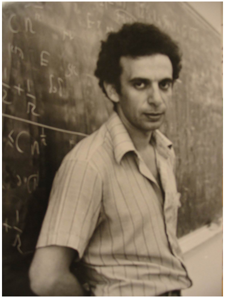


Problem solved by Witold Bednorz and Rafal Latała (2014).
Generalization to empirical processes.

Concentration of measure



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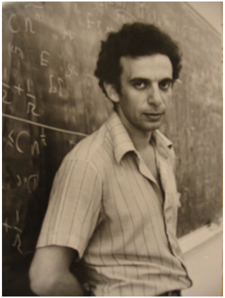

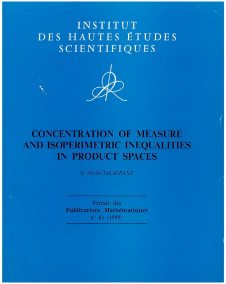

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Concentration of measure in product spaces $\rightsquigarrow$ Talagrand's convex concentration inequality (1995).
"*It is surely my most popular result, and the one most people will learn.*"

Some consequences of Talagrand's convex concentration inequality

For every set $A \subseteq \mathbb{R}^n$ and $t \geqslant 0$, we write

$$A_t := \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, A) \leqslant t\}.$$

Theorem (Michel Talagrand)

There exists $c > 0$ such that if $A \subseteq \mathbb{R}^n$ is convex and X uniformly distributed over $\{-1, 1\}^n$, then

$$\mathbb{P}[X \in A]\,\mathbb{P}[X \notin A_t] \leqslant \exp(-ct^2).$$

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For possibly non-convex A, this is false, c.f.

$$A := \{x \in \{-1,1\}^n \mid x \text{ has at least } \frac{n}{2} + \sqrt{n} \text{ ``+1'' coordinates}\}.$$

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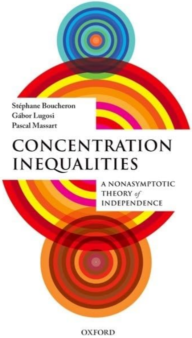
Theorem (Michel Talagrand)

There exists $c > 0$ such that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz, and X uniformly distributed over $\{-1,1\}^n$, then

$$\mathbb{P}\left[|f(X) - \mathbb{E}f(X)| \geqslant t\right] \leqslant 2\exp(-ct^2).$$

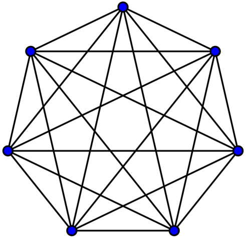
The entropy method

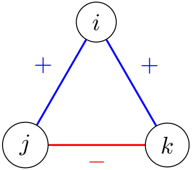
Michel Ledoux (1997, 1999, . . .) introduces the entropy method to prove concentration inequalities and recovers many results with more transparent proofs.



Stéphane Boucheron, Gábor Lugosi, Pascal Masssart (2009) + book (2013).

Spin glasses





Let (W_{ij}) be i.i.d. $\mathcal{N}(0,1)$, and set

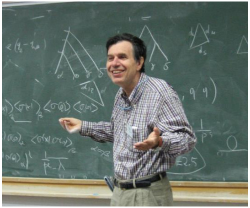
$$H_N(\sigma) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} W_{ij}\sigma_i\sigma_j.$$

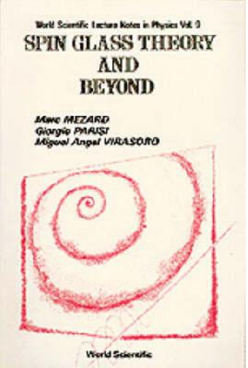
Large-N asymptotics of

$$\frac{1}{N} \max_{\sigma\in\{-1,1\}^N} H_N(\sigma) \qquad ?$$

$$\frac{1}{N} \log \sum_{\sigma\in\{-1,1\}^N} \exp(\beta H_N(\sigma)) \qquad ?$$

Parisi's formula



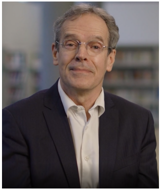


Giorgio Parisi (1979, 1980) predicts

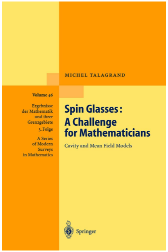
$$\frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\{\pm1\}^N}\exp\left(\beta H_N(\sigma)\right)\xrightarrow[N\to\infty]{}\inf_{\mu}\{\cdots\}.$$

Referee's report (as summarized by Parisi):
"The approach does not make sense, but the numbers coming from the formulae are reasonable, so it can be published."

Spin glasses



Erwin Bolthausen gets Talagrand interested ($\simeq$ 1995).
Regular meetings with Marc Mézard.



Spin glasses: a challenge to mathematicians (1998 paper, 2003 book).

"The goal was to make sure that no stone was left unturned."

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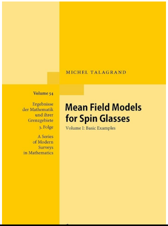
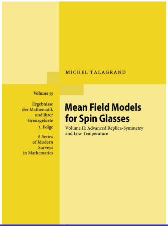
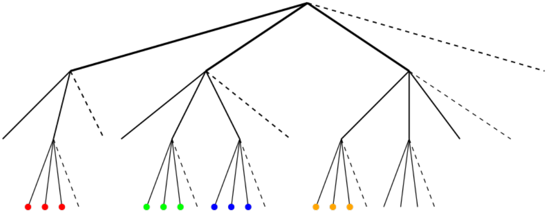
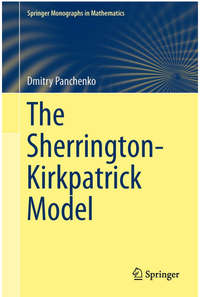




Ultrametricity of the Gibbs measure





Dmitry Panchenko (2013)

The Gibbs measure $\propto e^{\beta H_N(\sigma)}$

becomes ultrametric as $N \to +\infty$.

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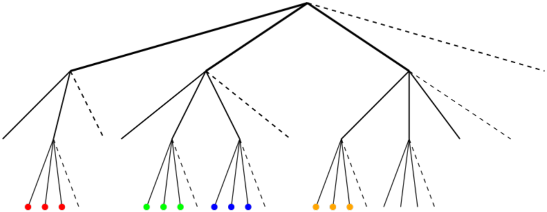

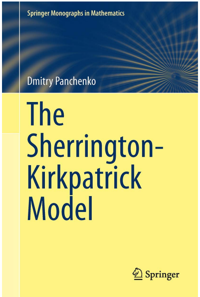

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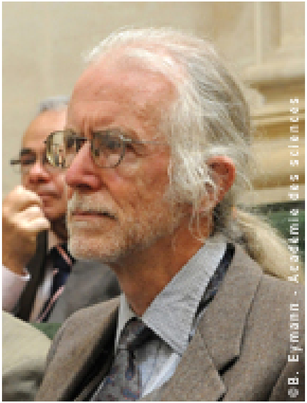
And a lot more now!

A personal note



"I would say that the main feature of my work style is that I try to get a full understanding. I cannot stop until I feel I completely understand the problem."