On random matrices

with complex eigen- values/vectors

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IHP, Paris, 2024

Plan

— (Very short) Physical motivation

1. Anderson & Many-Body Localization

— Three kinds of random matrix models

1. Rosenzweig-Porter GOE model
2. Weighted Erdos-Renyi graph
3. Rosenzweig-Porter Wishart model

— Conclusions

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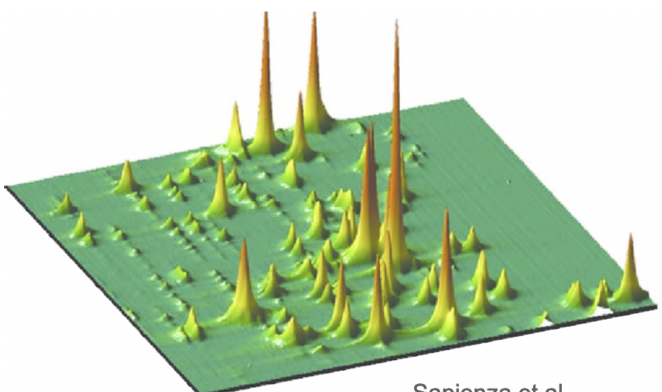
Anderson Localization

$$H = \sum_{ij} \underbrace{t_{ij}}_{j \to i \text{ hopping}} c_i^\dagger c_j + \sum_i \underbrace{\epsilon_i}_{\text{random energies}} \underbrace{c_i^\dagger c_i}_{\text{fermion number}}$$

i.i.d. random local energies from $\epsilon_i \in [-W/2, W/2]$

Competition between kinetic and random disordered terms

No interactions



Sapienza et al.,
Science 327, 1352 (2010)

Real space localization

$\psi(\vec{r}) \sim e^{-|\vec{r}-\vec{r}_o|/\xi}$

$d \leq 2$ for infinitesimal disorder

$d > 2$ metal/insulator transition at W_c

Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. 109, 1492 (1958)

Many-body Localization

From the real space Hamiltonian with interactions

$$H = \sum_{ij} t_{ij}\, c_i^\dagger c_j \;+\; \sum_i \epsilon_i\, c_i^\dagger c_i + \sum_{ij} \underbrace{V_{ij}\;\; c_i^\dagger c_i c_j^\dagger c_j}_{\text{fermion}-\text{fermion inter}}$$

to the Fock-space Hamiltonian

$$H' = \sum_{\alpha\neq\beta} \underbrace{T_{\alpha\beta}}_{\text{state change}} \; |\alpha\rangle\langle\beta| \;+\; \sum_{\alpha} \underbrace{E_\alpha}_{\text{energies}} \; |\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle = |n_1, n_2, \ldots, n_N\rangle$$

n_i occupation numbers

$T_{\alpha\beta}$ hierarchical

Fock space is a graph

Altshuler, Gefen, Kamenev & Levitov, *Quasiparticle Lifetime in a Finite System : A Nonperturbative Approach*, Phys. Rev. Lett. 78, 2803 (1997)

Many-body Localization

Fock-space Hamiltonian

$$H' = \sum_{\alpha \neq \beta} \underbrace{T_{\alpha\beta}}_{\text{state change}} |\alpha\rangle\langle\beta| + \sum_{\alpha} \underbrace{E_{\alpha}}_{\text{energies}} |\alpha\rangle\langle\alpha|$$

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Fock space is a graph

Localization in Fock space

$\downarrow$

Extended
Metal

Multi-fractal ?
Bad metal ?

Localized
Insulator

W

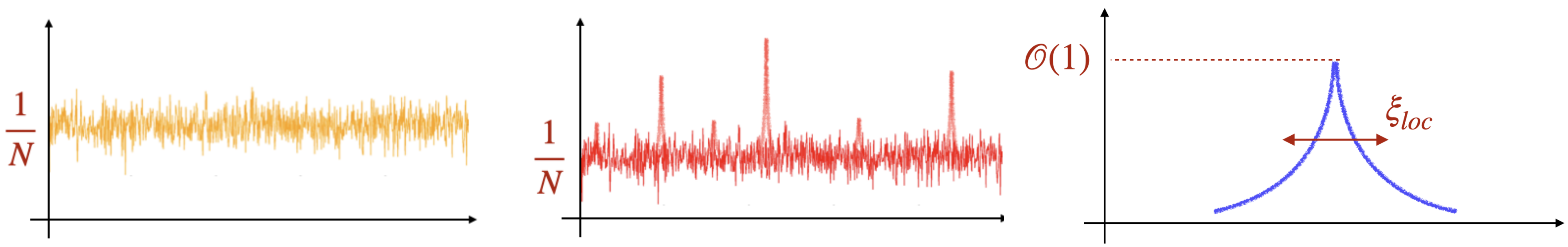
Basko, Aleiner & Altshuler, *Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states*, Ann. Phys. 321, 1126 (2006)

Many-body Localization

Multifractality

Extended	**Multi-fractal ?**	Localized	W
Metal	**Bad metal ?**	Insulator

$|\psi_i|^2$



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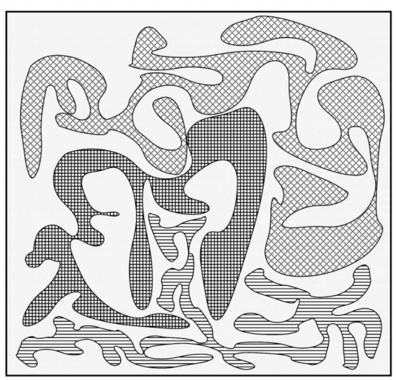


Random Matrix Model ?

Inverse participation ratio

$$I_q = \sum_i |\psi_i|^{2q} \propto N^{-(q-1)D_q}$$

$D_q = 0$ Localized

$D_q = 1$ Extended

$0 < D_q < 1$ Multifractal

Plan

— (Very short) Physical motivation

1. Anderson & Many-Body Localization
2. Neural Networks[1]
3. Pure random matrix theory[2]

— Three kinds of random matrix models

1. Rosenzweig-Porter GOE model
2. Weighted Erdos-Renyi graph
3. Rosenzweig-Porter Wishart model

— Conclusions

[1] **P. Gong et al**, **S. Nechaev et al**, several papers ; [2] **P Akara-pipattana & O Evnin**, several papers.

Two kinds of random matrices

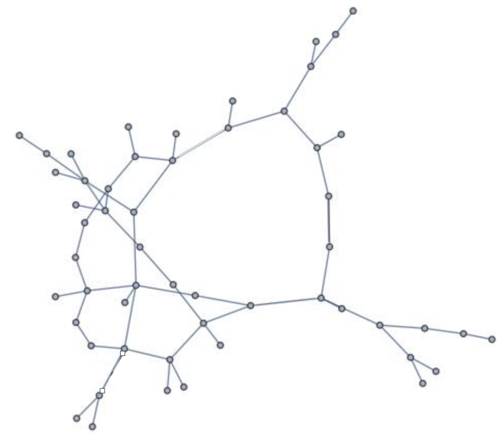
- **Rosenzweig-Porter (RP)**

  Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}}\,\mathbb{B}$$

- **Weighted Erdös-Rényi graph**

  Fluctuating connectivity & random hops



Motivation

Search of random matrices with (multi)fractal eigenvectors

Venturelli, LFC, Schehr, Tarzia, *Replica approach to the generalized Rosenzweig-Porter model*, SciPost Phys. 14, 110 (2023)

LFC, Schehr, Tarzia, Venturelli, *Multifractal phase in the adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

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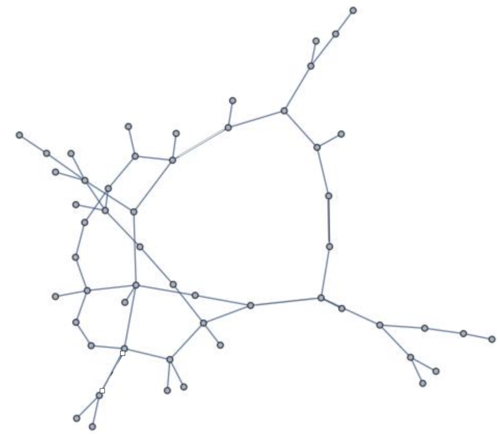

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The RP Model

Sum of random matrices

Take a diagonal $N \times N$ matrix $\mathbb{A}$ with *i.i.d.* real elements $a_i = \mathcal{O}(1)$ taken from a $p_a(a_i)$

$$\mathbb{A} = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & \dots & 0 & a_{N-1\,N-1} & 0 \\ 0 & \dots & 0 & 0 & a_{N\,N} \end{pmatrix}$$

and a real symmetric $N \times N$ matrix $\mathbb{B}$ from the Gaussian Orthogonal Ensemble $2\mathbb{B} = \mathbb{C} + \mathbb{C}^T$

$$\mathbb{B} = \begin{pmatrix} b_{11} & b_{12} & & \dots & b_{1\,N} \\ b_{12} & b_{22} & b_{23} & \dots & b_{2\,N} \\ & \dots & \dots & \dots & \\ b_{1\,N-1} & \dots & \dots & b_{N-1\,N-1} & b_{N-1\,N} \\ b_{1\,N} & & \dots & b_{N-1\,N} & b_{N\,N} \end{pmatrix}$$

with $b_{ij} = \mathcal{O}(1) \in \mathbb{R}$ taken from $p_b(b_{i\neq j}) \propto e^{-b_{ij}^2/2}$ and $p_b(b_{ii}) \propto e^{-b_{ii}^2/4}$

The RP Model

Sum of random matrices

Add them in the form

$$\mathbb{H} = \mathbb{A} + \underbrace{\frac{\nu}{N^{\gamma/2}} \mathbb{B}}_{\gamma>1 \text{ “perturbation”}}$$

with ν and γ two $\mathbb{R}$ parameters $\mathcal{O}(1)$

& N the size of the square matrices

Initial **motivation**: adapt random matrix theory to atomic physics studies

Rosenzweig & Porter, *Repulsion of Energy Levels in Complex Atomic Spectra*, Phys. Rev. 120, 1698 (1960)

More recently: many-body localization and the hypothetical bad metal phase

Kravtsov, Khaymovich, Cuevas & Amini, *A random matrix model with localization and ergodic transitions*, New. J. Phys. 17, 122002 (2015)

Sum of random matrices & applications - free probability

A. Zee, *Law of addition in random matrix theory*, Nucl. Phys. B 474, 726 (1996)

Density of eigenvalues

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\,\mathbb{B}$

Averaged spectral density $\quad \rho_N(\lambda) \equiv \left[\frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right]_{\mathbb{H}}$

λ_i the eigenvalues

Limits

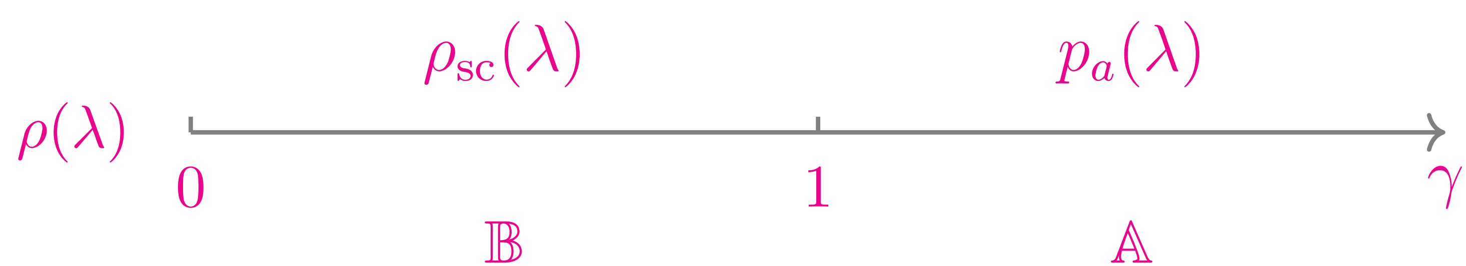
– For $\gamma \gg 1$ the effect of $\mathbb{B}$ is negligible and

$\rho_N(\lambda)$ is just $p_a(\lambda)$

– In the Gaussian Orthogonal Ensemble (GOE), $\gamma = 1$ and $p_a(a_{ii}) = \delta(a_{ii})$, only $\mathbb{B}$ counts, and

$\rho(\lambda) \equiv \lim_{N\to\infty} \rho_N(\lambda)$ is the semi-circle law $\rho_{\rm sc}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$

In general



Level spacings

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\, \mathbb{B}$

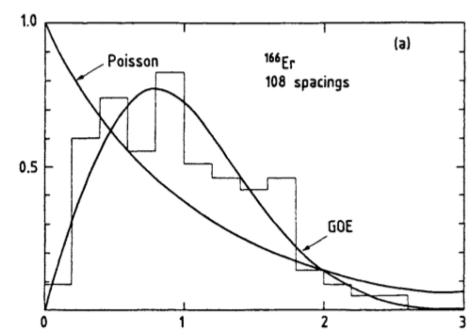
Level statistics The level spacings $s_i = \lambda_{i+1} - \lambda_i$, normalized by their mean $\langle s_i \rangle$, are distributed according to

Limits

– for independent levels, Poisson's p.d.f. $p(s) = e^{-s}$

– in the GOE case, the Wigner surmise

$$p(s) = \frac{\pi}{2}\, s\, e^{-\pi s^2/4}$$



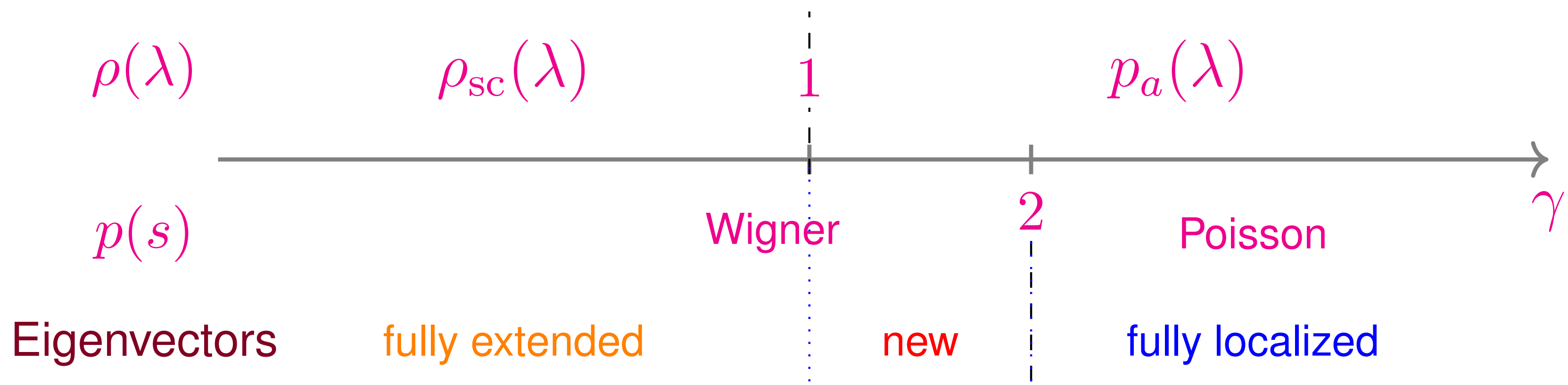
Wigner (GOE) Poisson

$p(s)$ 0 2 γ

$\mathbb{B}$ $\mathbb{A}$

Eigenvectors

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\,\mathbb{B}$



– For $\gamma > 2$, $\mathbb{H} \sim \mathbb{A}$ and the eigenvectors (wave functions) ψ are fully localized

$n = \mathcal{O}(1)$ non-zero component

– For $\gamma < 1$, $\mathbb{H} \sim \nu N^{-\gamma/2}\,\mathbb{B}$ and the eigenvectors ψ are fully extended

$n = \mathcal{O}(N)$ non-zero components

– For $1 < \gamma < 2$, the eigenvectors ψ are localized over a fractal number of sites

$1 \ll n = \mathcal{O}(N^{2-\gamma}) \ll N$ non-zero components

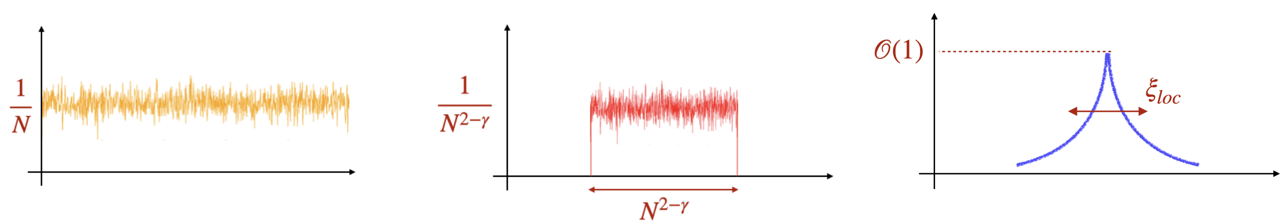
Kravtsov, Khaymovich, Cuevas & Amini, *A random matrix model with localization and ergodic transitions*, New. J. Phys. 17, 122002 (2015)

Eigenvectors

Properties of $\mathbb{H} = \mathbb{A} + \nu N^{-\gamma/2}\, \mathbb{B}$

$\rho(\lambda)$	$\rho_{\rm sc}(\lambda)$		$p_a(\lambda)$
γ axis	$\gamma < 1$	$1 < \gamma < 2$	$\gamma > 2$
Eigenvectors	fully extended	fractal	fully localized
	metal	bad metal ?	insulator
	ergodic	non-ergodic de-localized	non-ergodic

$|\psi_i^{(\alpha)}|^2$



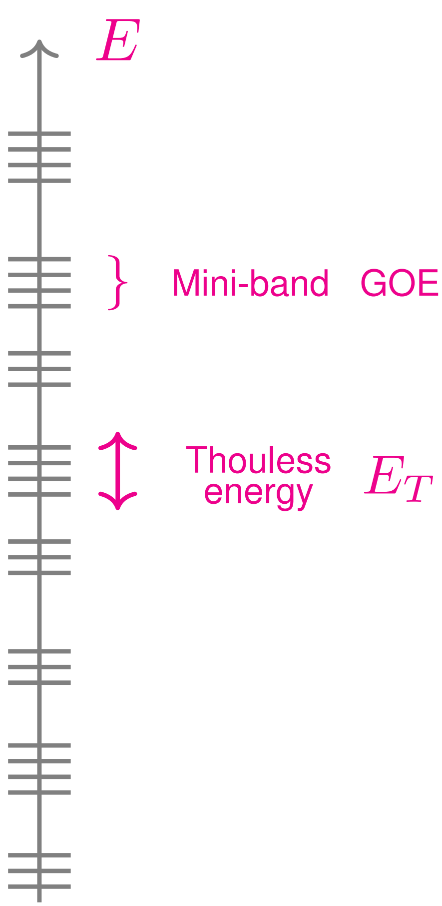
Horizontal axis: “site” i index ordered according to $a_1 \le \cdots \le a_N$

Kutlin & Khaymovich, *Anatomy of the eigenstates distribution...* SciPost Phys. 16, 008 (2024)

de Tomasi, Amini, Bera, Khaymovich & Kravtsov, *Survival probability in Generalized Rosenzweig-Porter random matrix ensemble*, SciPost Phys. 6, 014 (2019) $R(t) \equiv [|\langle\psi(t)|\psi(0)\rangle|^2]_{\mathbb{H}} \xrightarrow[t\to\infty]{} N^{-D_f}$

Picture

Mini-bands and the fractal dimension for $1 < \gamma < 2$



Perturbation theory $\Rightarrow \lambda_i \sim a_i + 4\zeta \frac{1}{N} \sum_{j(\neq i)} \frac{b_{ij}^2}{a_i - a_j}$

Average spreading of the eigenvalues $\Rightarrow$ **Thouless energy**

$$E_T \equiv \Big[|\lambda_i - a_i| \Big]_{\mathbb{H}} \sim \zeta \equiv (\nu^2/4)\, N^{1-\gamma} \gg 1/N$$

width of the **mini-bands** with GOE statistics

The number of eigenvectors “hybridized" by the perturbation

$$\# \sim \frac{E_T}{(N\rho(\lambda))^{-1}} \sim N^{D_f} \quad \text{with} \quad \boxed{D_f = 2 - \gamma < 1}$$

support of the eigenvectors of the perturbed matrix $\mathbb{H}$

But no evidence for **multifractality** in this model

Methods & our results

More details on the eigenvalues

The **resolvent**

$$G_N(z) \equiv \frac{1}{N}\,\mathrm{Tr}\,(z\mathbb{I} - \mathbb{H})^{-1} = \frac{1}{N}\sum_i \frac{1}{z-\lambda_i} \qquad \text{with } \; z \in \mathbb{C}$$

N poles at $z = \lambda_i$ the real eigenvalues of $\mathbb{H}$

Average over $\mathbb{H}$ and take the large N limit

$$\lim_{N\to\infty} [G_N(z)]_{\mathbb{H}} = \int d\lambda' \, \frac{\rho(\lambda')}{z-\lambda'}$$

which is the Stieltjes transform of the av. density of eigenvalues ρ with inverse

$$\boxed{\rho(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \lim_{N\to\infty} [G_N(\lambda - \mathrm{i}\eta)]_{\mathbb{H}}}$$

Methods & our results

Replica trick

$$\rho_N(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \left[\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda - \mathrm{i}\eta - \lambda_i} \right]_{\mathbb{H}}$$

$$= -\frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \frac{\partial}{\partial\lambda} \left[\ln \mathcal{Z}_N(\lambda - \mathrm{i}\eta) \right]_{\mathbb{H}}$$

with the partition function $\mathcal{Z}_N(z) = \frac{1}{(2\pi \mathrm{i})^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N r \; e^{-\frac{1}{2}\mathbf{r}^{\mathrm{T}}(z\mathbb{I}-\mathbb{H})\,\mathbf{r}}$

Using replicas $[\ln \mathcal{Z}_N]_{\mathbb{H}} = \lim_{n\to 0} \frac{1}{n} \ln [\mathcal{Z}_N^n]_{\mathbb{H}}$ & saddle-point for $N \to \infty$

– usual replica symmetric Ansatz on $NQ_{ab} = \langle \mathbf{r}^a \cdot \mathbf{r}^b \rangle$ (difficult) or

– **rotationally invariant Ansatz** in replica space for the density (simpler !)

$$\mu(\vec{r}) = \mu(r^1, \ldots, r^a) = \frac{1}{N} \sum_{i=1}^{N} \prod_{a=1}^{n} \delta(r^a - r_i^a) = \overline{\mu}(r)$$

Edwards & Jones, *The eigenvalue spectrum of a large symmetric random matrix*, J. Phys. A 9, 1595 (1976)
Livan, Novaes & Vivo, *Introduction to Random Matrices – Theory and Practice*, arXiv : 1712.07903, SpringerBriefs in Mathematical Physics 26 (2018)

The Zee formula

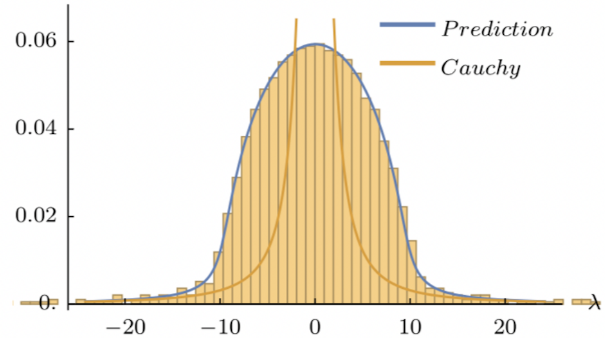
The averaged density of eigenvalues

After some lengthy but simple steps

$$\rho_N(\lambda) = -\frac{1}{\pi\zeta} \lim_{\eta\to 0^+} \mathrm{Re}\, C(\lambda - \mathrm{i}\eta)$$
$$C(\lambda) = \mathrm{i}\zeta\, G_a(\lambda + 2\mathrm{i}\, C(\lambda))$$

$NG_a(z) = \mathrm{Tr}(\mathbb{A} - z\mathbb{I})^{-1}$ the global resolvent of $\mathbb{A}$

and $\zeta = \frac{\nu^2}{4} N^{1-\gamma}$



Solution for Cauchy p_a

$N = 2000, \gamma = 1.1$

$\nu = 10, \zeta = 11.7$

Evaluate numerically, leading finite size corrections captured &

approximate analytic expression for $\rho_N(\lambda)$ in the limit $\zeta \ll 1$ and any p_a

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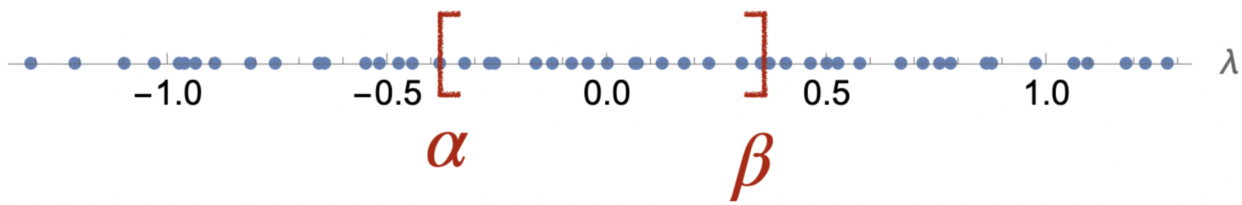
Generalization of the **Zee formula** in *Law of addition in random matrix theory*, Nucl. Phys. B 474, 726 (1996)

Krajenbrink, Le Doussal & O'Connell, *Tilted elastic lines with columnar and point disorder, non-Hermitian quantum mechanics, and spiked random matrices: pinning and localization*, Phys. Rev. E 103, 042120 (2021)

The level compressibility



$I_N(\alpha,\beta) = N \int_\alpha^\beta d\lambda\, \rho_N(\lambda)$ counts how many eigenvalues fall in the interval

Large deviation function $\mathcal{F}_{[-E,E]}(s) = \lim_{N\to\infty} \frac{1}{N} \ln \left[e^{-sI_N(-E,E)} \right]_{\mathbb{H}}$ can be calculated with the **replica method** and then get the moments $\left[I^k(-E,E) \right]_{\mathbb{H}}^c$

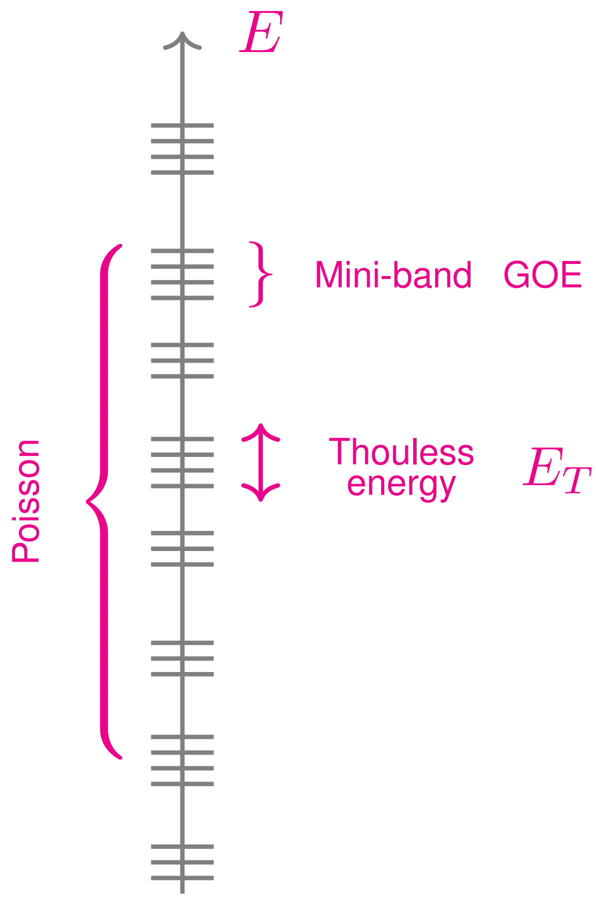
and the **level compressibility** $$\chi(E) = \frac{\left[I^2(-E,E) \right]_{\mathbb{H}}^c}{\left[I(-E,E) \right]_{\mathbb{H}}}$$

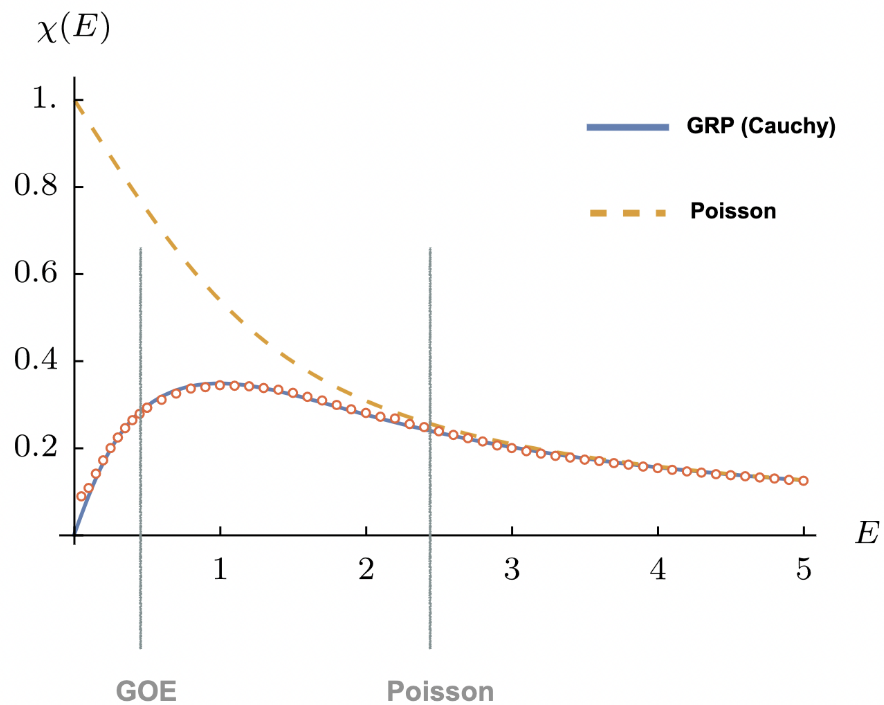
Metz & Pérez Castillo, *Large Deviation Function for the Number of Eigenvalues of Sparse Random Graphs Inside an Interval*, Phys. Rev. Lett. 117, 104101 (2016)

Metz, *Replica-symmetric approach to the typical eigenvalue fluctuations of Gaussian random matrices*, J. Phys. A 50, 495002 (2017)

Picture recovered

Mini-bands in the intermediate $1 < \gamma < 2$ regime





Within the bands

Well beyond the bands

$$\chi(E) = \frac{\left[I^2(-E,E)\right]^c_{\mathbb{H}}}{\left[I(-E,E)\right]_{\mathbb{H}}}$$

Picture recovered

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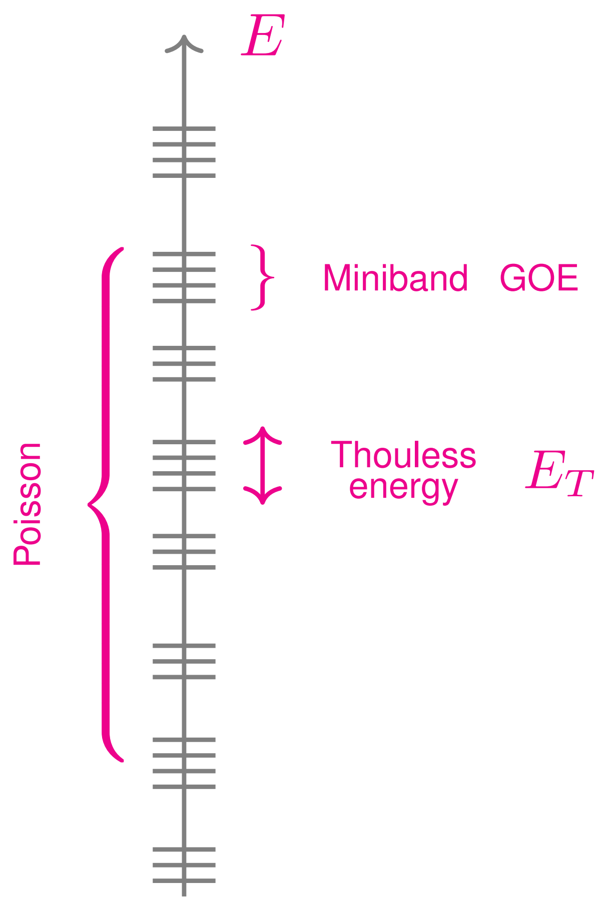
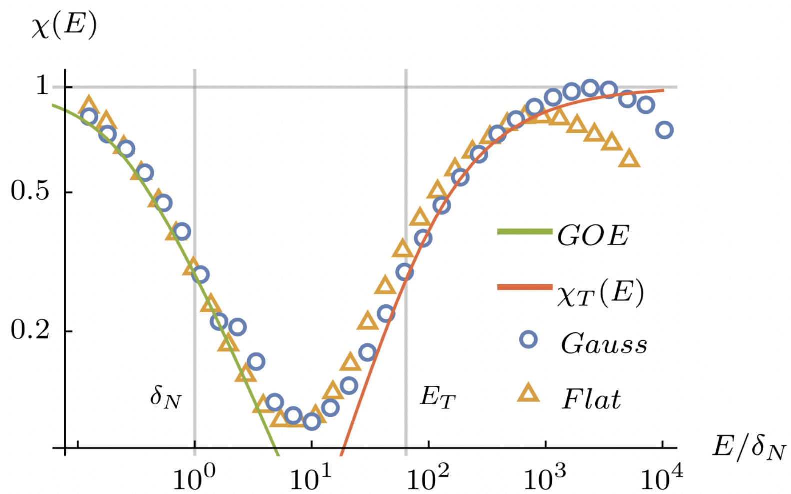


Zoom over small E



δ_N mean level spacing

$E \sim E_T$ Scaling limit $y = \frac{1}{2\pi p_a(0)} \frac{E}{E_T}$

$$\tilde{\chi}(y) = \frac{1}{\pi y} \left[2y \arctan y - \ln(1 + y^2) \right]$$

Results: it is universal with respect to p_a

Results so far

Rosenzweig Porter GOE model

— We rederived $\rho(\lambda)$ using replicas and a rotational symm. *Ansatz*

— We derived the Zee formula in a simple way
(with finite size dependencies)

— We obtained the level compressibility

— It satisfies a scaling form which is universal with respect to p_a

— Is it also universal with respect to p_b? RP-Wishart, in progress

Two kinds of random matrices

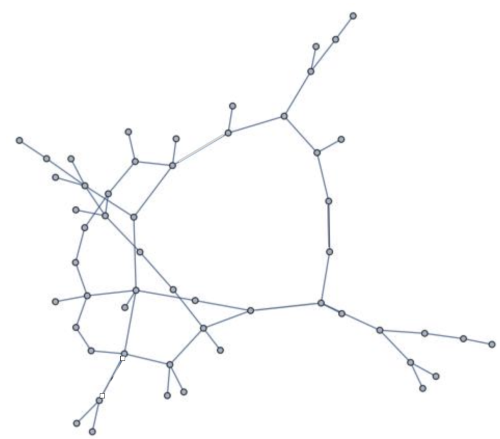
- Rosenzweig-Porter (RP)

  Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}} \mathbb{B}$$

- **Weighted Erdös-Rényi graph**

  Fluctuating connectivity & random hops



Motivation

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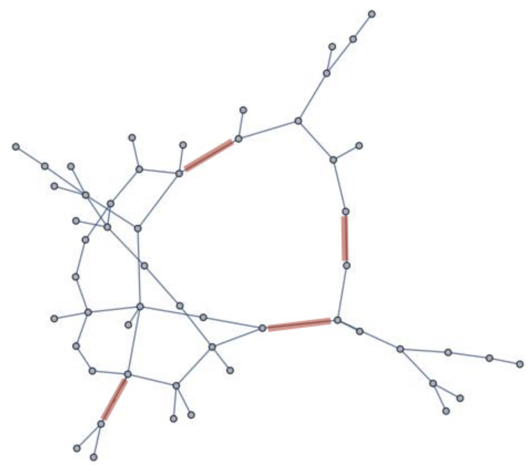
LFC, Schehr, Tarzia, Venturelli, *Multifractal phase in the adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Weighted random graph

Erdös-Rényi

Random graph | Hopping

$$H_{ij} = \frac{1}{\sqrt{p}}\,\sigma_{ij} t_{ij} \qquad \sigma_{ij} = \begin{cases} 1 & \text{prob} = p/N \\ 0 & \text{prob} = 1 - p/N \end{cases} \qquad t_{ij} \;\; \text{GOE}$$



A sketch with $p = 3$

$p \geq 1$: a giant component with $\tilde{N} \propto N$ sites and $\mathcal{O}(N)$ finite size clusters.

Focus on the giant component only $\tilde{p} \propto p$

Note that the red links can have $t_{ij} \sim 0$

For the random graph: **Rodgers & Bray**, *Density of states of a sparse random matrix*, Phys. Rev. B 37, 3557 (1988). **Semerjian & LFC**, *Sparse random matrices : the eigenvalue spectrum revisited*, J. Phys. A 35, 4837 (2002). **Kuhn**, *Spectra of sparse random matrices*, J. Phys. A 41, 295002 (2008).

Eigenvectors

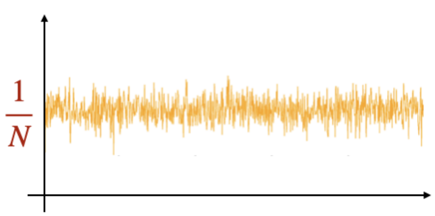
Inverse participation ratios & fractal dimensions

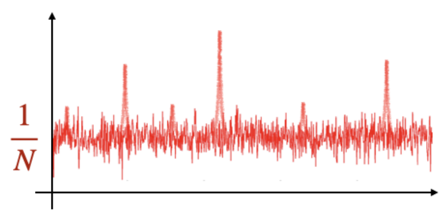
Take the αth eigenvector $\psi_i^{(\alpha)}$ with $i = 1, \ldots, N$ at a given energy E and calculate the disorder average IPR $\Rightarrow$ Fractal dimensions D_q

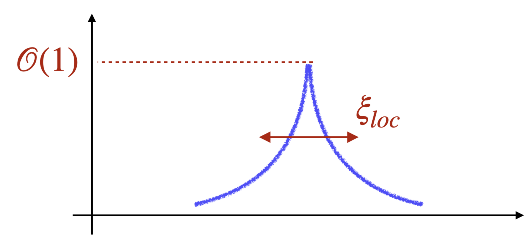
$$I_q = \left[\sum_{i=1}^{N} |\psi_i^{(\alpha)}|^{2q} \right]_{\mathbb{H}} \propto N^{(1-q)D_q} \qquad \text{(IPR)}$$

$D_q = 1 \ \forall \, q$	$D_q \neq 1, 0$	$D_q = 0 \ \forall \, q$
Extended	Special	Localized

$|\psi_i^{(\alpha)}|^2$







Method & our results

Cavity

The trace of the **resolvent matrix** $\mathbb{G}_N(z) = (\mathbb{H} - z\mathbb{I}_N)^{-1}$

$$G_N(z) \equiv \mathrm{Tr}\,\mathbb{G}_N(z) = \sum_{i=1}^{N} (\lambda_i - z)^{-1} \qquad \Rightarrow \qquad \rho_N(\lambda)$$

The **cavity Green's function** is the diagonal element on node i of the resolvent of the Hamiltonian $\mathbb{H}^{(j)}$ with its neighbor j removed

$$G_{i\to j}(z) = (\mathbb{H}^{(j)} - z\mathbb{I}_{N-1})^{-1}_{ii}$$

It satisfies the recursion relation

$$G_{i\to j}(z) = \Big(H_{ii} - z - \sum_{m\in\partial i\setminus j} H_{mi}^2\, G_{m\to i}(z) \Big)^{-1}$$

and the solution is used as an estimate of the diagonal elements $G_{ii}(z)$

which yield the IPR

$$\boxed{I_q(E) \propto \lim_{\eta\to 0^+} \eta^{q-1} \frac{1}{N} \sum_i |G_{ii}(z)|^q}$$

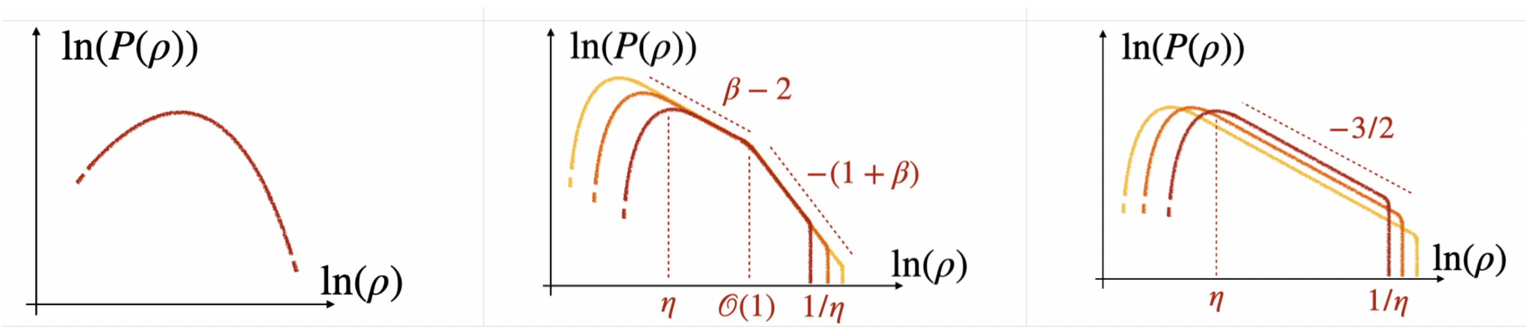
Local density of states

Definition & properties

$$\rho_i(E) \equiv \sum_\alpha |\psi_i^{(\alpha)}|^2\, \delta(E-\lambda_\alpha) = \frac{1}{\pi}\lim_{\eta\to 0^+} \mathrm{Im} G_{ii}(z)$$

$$P(\rho, z) = \left[\frac{1}{N}\sum_{i=1}^{N} \delta(\rho - \pi^{-1}\mathrm{Im}G_{ii}(z))\right]_{\mathbb{H}}$$

Extended | Special | Localized



Symmetry $P(\rho) = \rho^{-3} P(1/\rho)$ respected

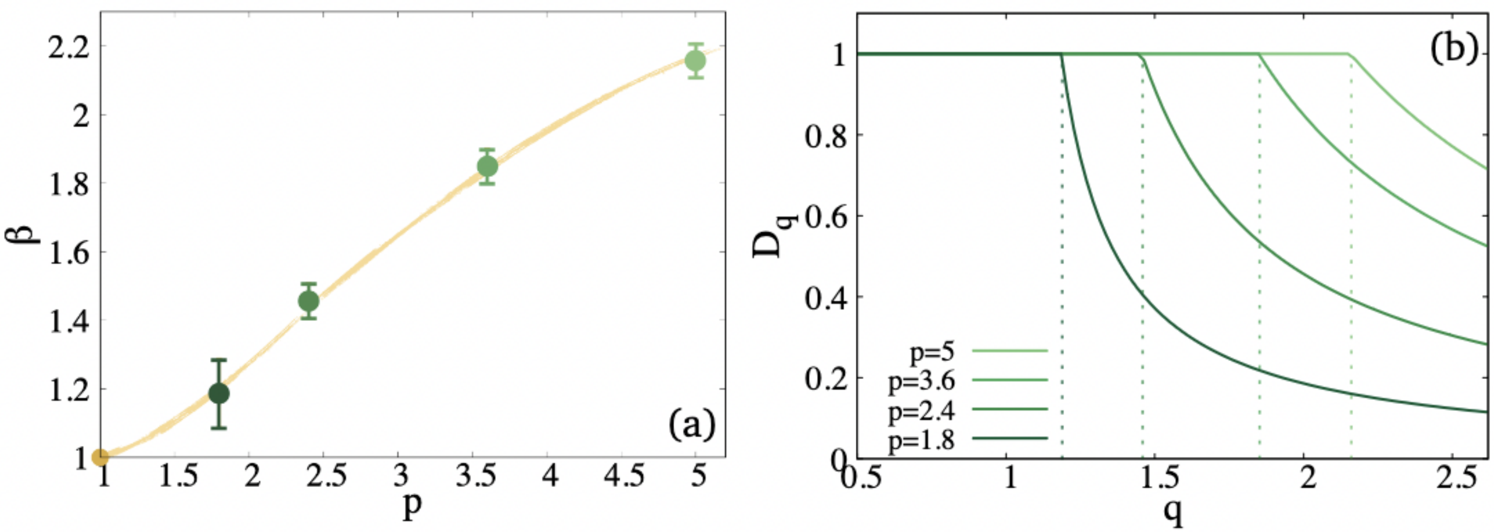
Mirlin, Fyodorov, Mildenberger & Evers, *Exact Relations between Multifractal Exponents at the Anderson Transition*, Phys. Rev. Lett. 97, 046803 (2006)

Fractal dimensions

In the special regime

From the analysis of the tail obtained with the cavity method

$$P(\rho) \sim \rho^{-(1+\beta)} \quad \Rightarrow \quad D_q = \begin{cases} \dfrac{\beta - 1}{q - 1} & q \geq \beta \\ 1 & q < \beta \end{cases}$$



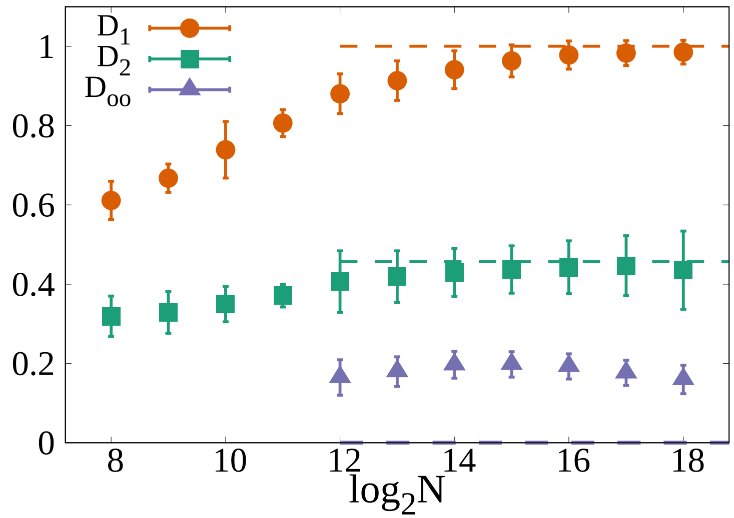
As p increases the graph becomes more and more connected, β increases, and the GOE fully extended behavior is approached with $D_q \to 1$ until larger and larger q

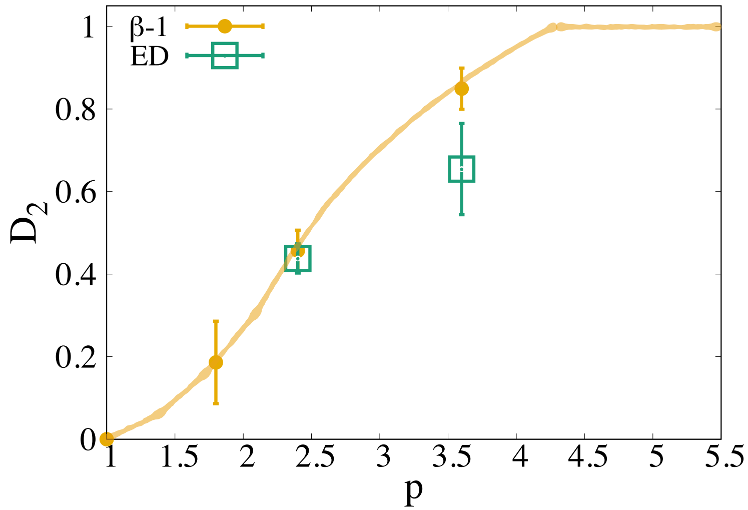
Fractal dimension

Cavity method vs. exact diagonalization

Cavity method vs. exact diagonalization

Horizontal dashed line vs. data points

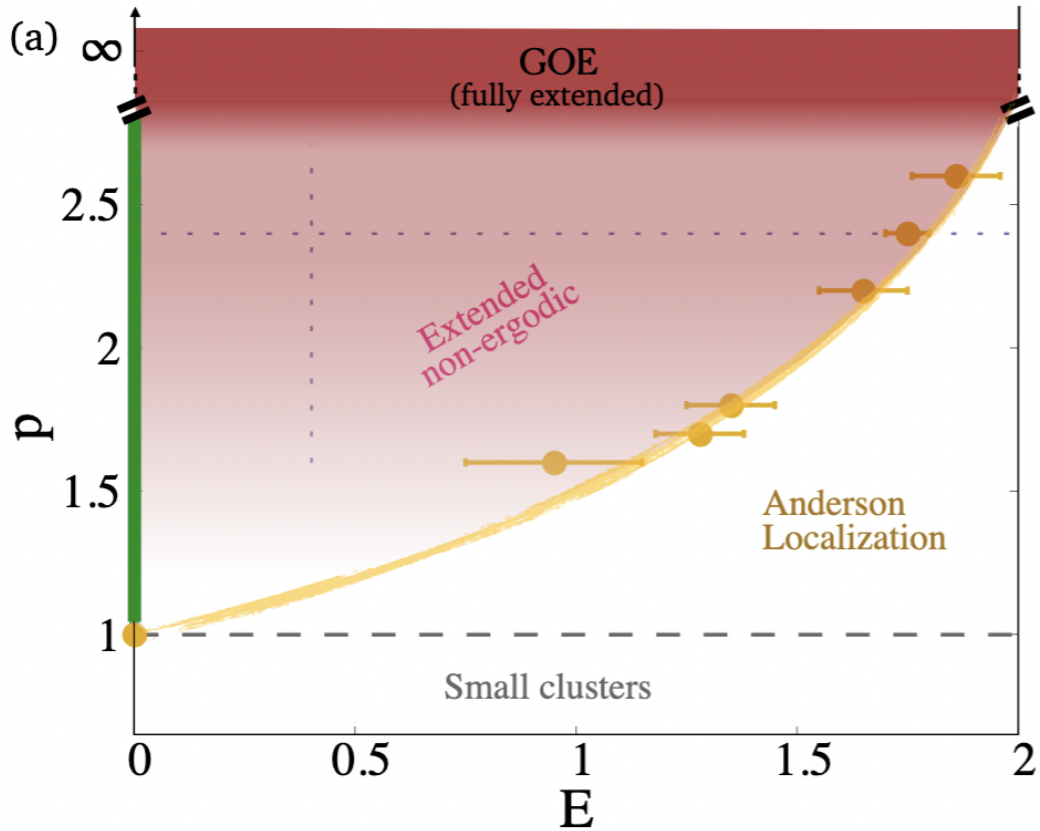




$p = 2.4, \quad E = 0.4, \quad \beta \sim 1.5$

Weighted random graph

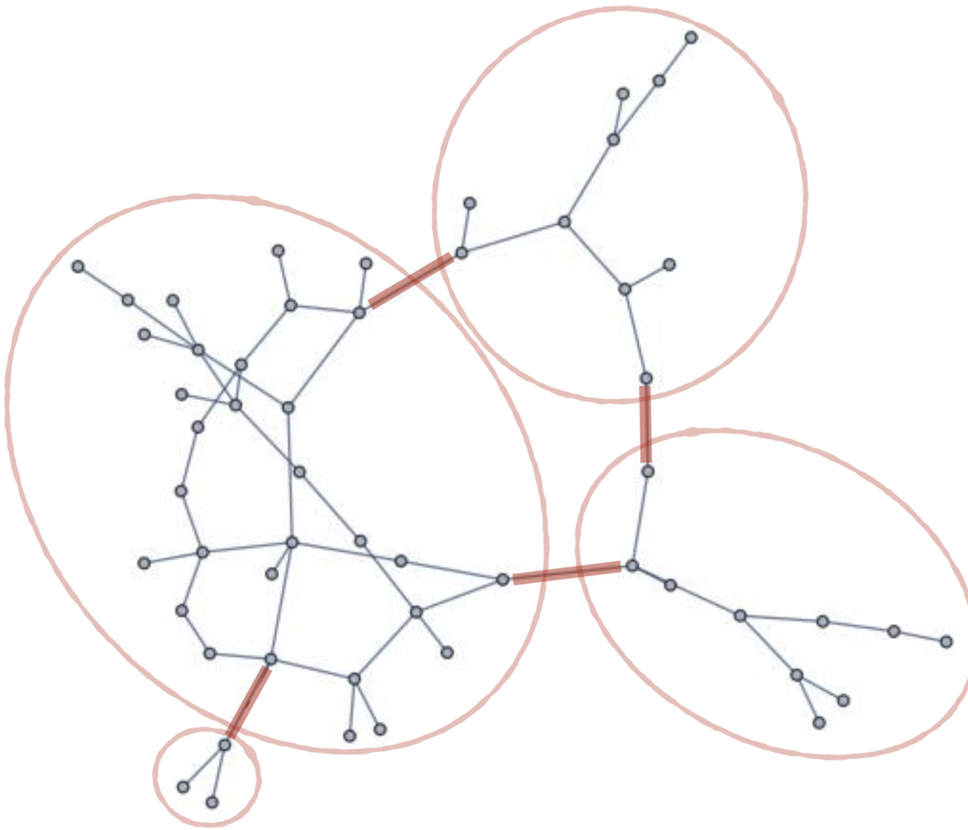
The phase diagram



LFC, Schehr, Tarzia & Venturelli, *Multifractal phase in the weighted adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Multifractal phase

Mechanism

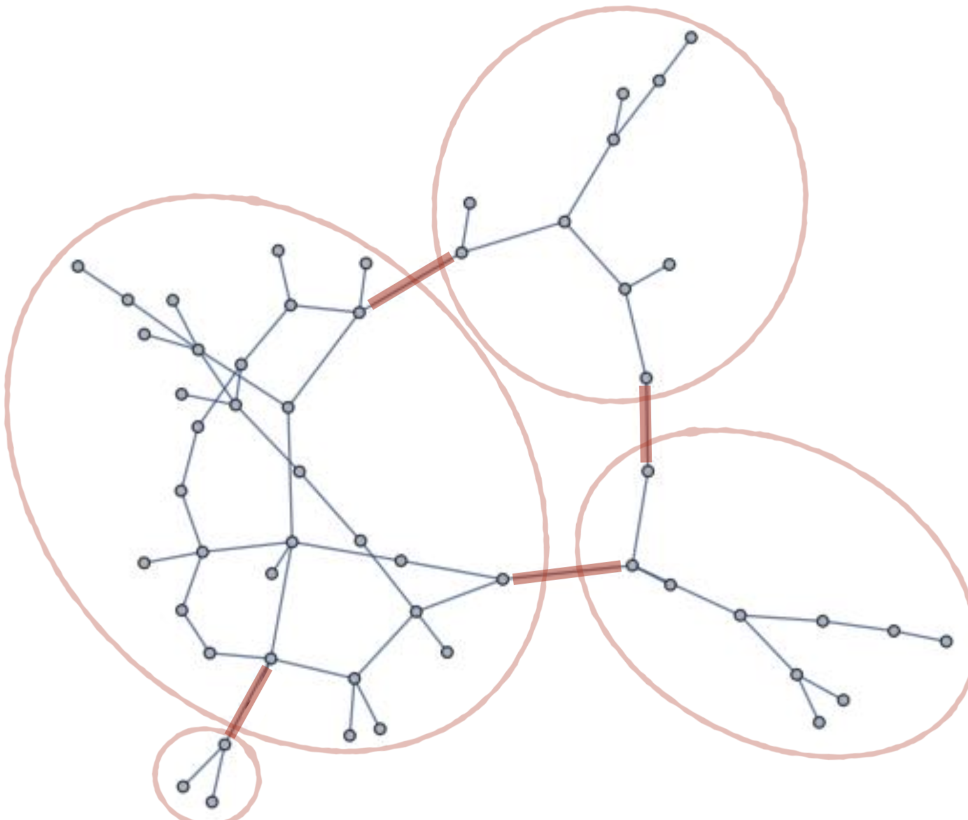


Graph heterogeneity $\Rightarrow$

effective fragmentation due to very weak links

Multifractal phase

Robustness



Re-wiring $\Rightarrow$ eliminate rooted trees

Re-weighting $\Rightarrow$ re-draw $h_{ij} < \nu$ into $h_{ij} > \nu$

Two kinds of random matrices

- **Rosenzweig-Porter (RP)**

Sum of random matrices

$$\mathbb{H} = \mathbb{A} + \frac{\nu}{N^{\gamma/2}} \mathbb{B}$$

$$1 < \gamma < 2$$

Properties

$$\rho(\lambda) = p_a(\lambda)$$

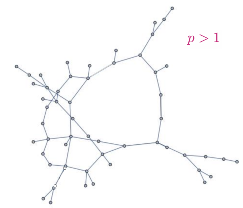
$$p(s) \propto s e^{-\pi s^2/4}$$

$$|\psi_i^{(\alpha)}|^2 \propto N^{D_2}$$

$$D_2 = \gamma - 2$$

- **Weighted Erdös-Rényi graph**

Fluctuating connectivity & random hops



$$\left[|\psi_i^{(\alpha)}|^{2q} \right]_{\mathbb{H}} \propto N^{(1-q)D_q}$$

$$D_q = \begin{cases} \frac{\beta - 1}{q - 1} & q \geq \beta(p, E) \\ 1 & q < \beta(p, E) \end{cases}$$

Multifractal phase

Venturelli, LFC, Schehr, Tarzia, *Replica approach to the generalized Rosenzweig-Porter model*, SciPost Phys. 14, 110 (2023)

LFC, Schehr, Tarzia, Venturelli, *Multifractal phase in the adjacency matrices of random Erdös-Rényi graphs*, Phys. Rev. B 110, 174202 (2024)

Work in progress

Wishart matrices

Universality of χ with respect to $\mathbb{B}$?

Take a rectangular $N \times M$ random matrix $\mathbb{D}$ with $p_d(d_{ij}) \propto e^{-d_{ij}^2/2}$

(Ginibre ensemble)

Build the square $N \times N$ symmetric random matrix $\mathbb{W} = M^{-1}\, \mathbb{D}\mathbb{D}^T$

Could be an estimate of a correlation matrix

The averaged density of eigenvalues of $\mathbb{W}$ is the **Marcenko-Pastur** law

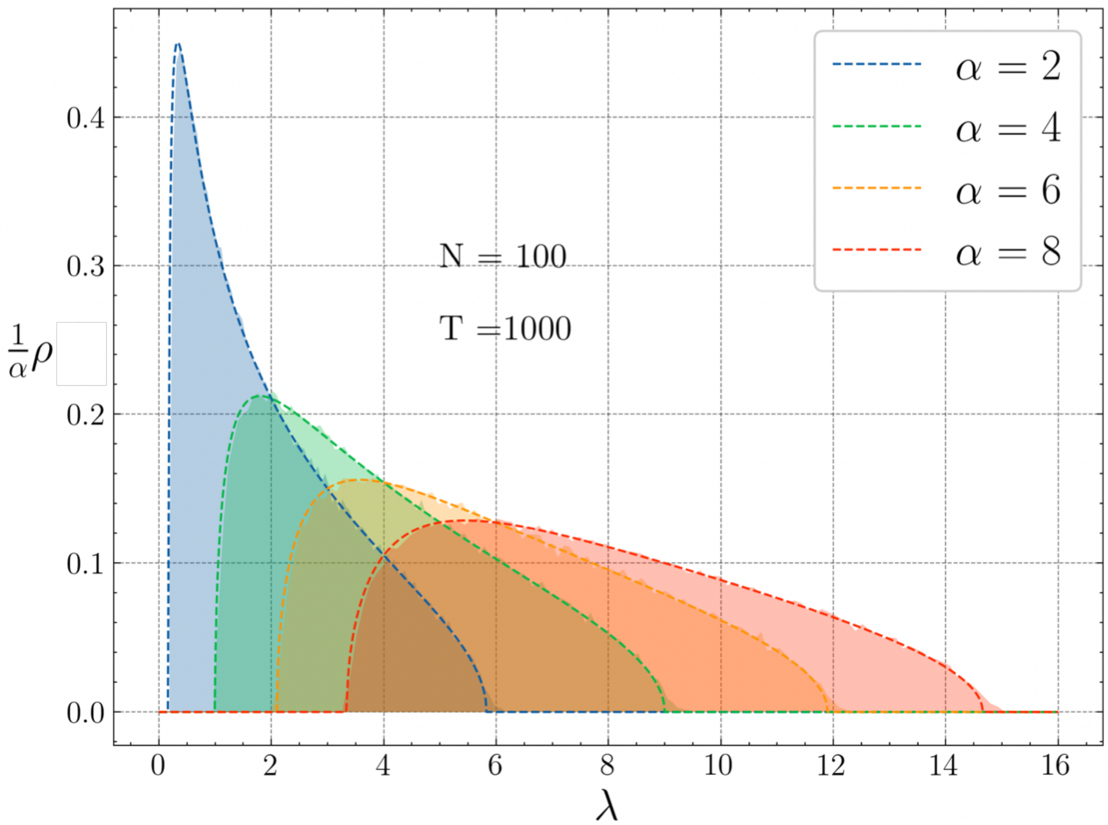
$$\rho_{\rm MP}(\lambda) = \alpha \frac{\sqrt{(\lambda - \xi_+)(\xi_- - \lambda)}}{2\pi\lambda} + (1-\alpha)\delta(\lambda)\theta(1-\alpha)$$

with $\xi_\pm \equiv \left(1 \pm \alpha^{-1/2}\right)^2$ and $\alpha \equiv M/N$

Wishart *The generalised product moment distribution in samples from a normal multivariate population*, Biometrika, 20A, 32 (1928). **Marchenko & Pastur**, *Distribution of eigenvalues for some sets of random matrices*, Matematicheskii Sbornik 114, 507 (1967). **Zavatone-Veth & Cengiz Pehlevan**, *Replica method for eigenvalues of real Wishart product matrices*, arXIv:2209.10499

Wishart matrices

Density of eigenvalues



Numerics vs. analytic expression

RP with Wishart matrices

Questions

Build the Wishart-Rosenzweig-Porter model $\mathbb{A} + \frac{\nu}{N^{1-\gamma}}\, \mathbb{W}$

— What is the composition rule for the av. density of eigenvalues ? Extension of the Zee formula.

— Is the level compressibility χ following the same universal law as for the RP - GOE model ?

In collaboration with **V. Delapalme, D. Venturelli & M. Tarzia**

Multifractal phase

Return probability & wave function overlap

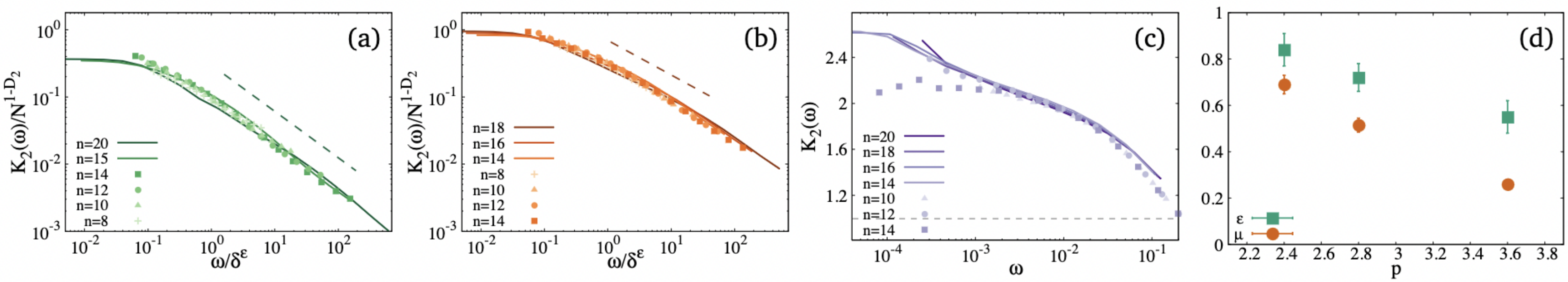


FIG. 10. Overlap correlation function $K_2(\omega;E)$, see Eq. (36), for $E = 0.4$ and $p = 2.4$ (a), $p = 2.8$ (b), and $p = 5$ (c). In panels (a) and (b) the energy separation ω is rescaled by the Thouless energy δ^{ϵ} — with a p-dependent exponent $\epsilon \in [0,1]$ — and the y-axis is rescaled by N^{1-D_2} — where $D_2 = \beta - 1$, with β given in Fig. 6(a). Continuous lines correspond to the results obtained by solving the self-consistent cavity equations, while symbols show the exact diagonalization results. The dashed lines in panels (a) and (b) represent the power-law decay of correlations for $\omega \gg \delta^{\epsilon}$ as $K_2(\omega) \propto (\omega/\delta^{\epsilon})^{-\mu}$, with a p-dependent exponent μ. The horizontal gray line in panel (c) corresponds to the Wigner-Dyson behavior for GOE matrices, $K_2(\omega) = 1$. In panel (d) we plot our numerical estimates for the exponents ϵ (squares) and μ (circles) varying the average degree p, showing that a standard fully-delocalized behavior (with $\epsilon, \mu \to 0$) is progressively reached upon increasing p.

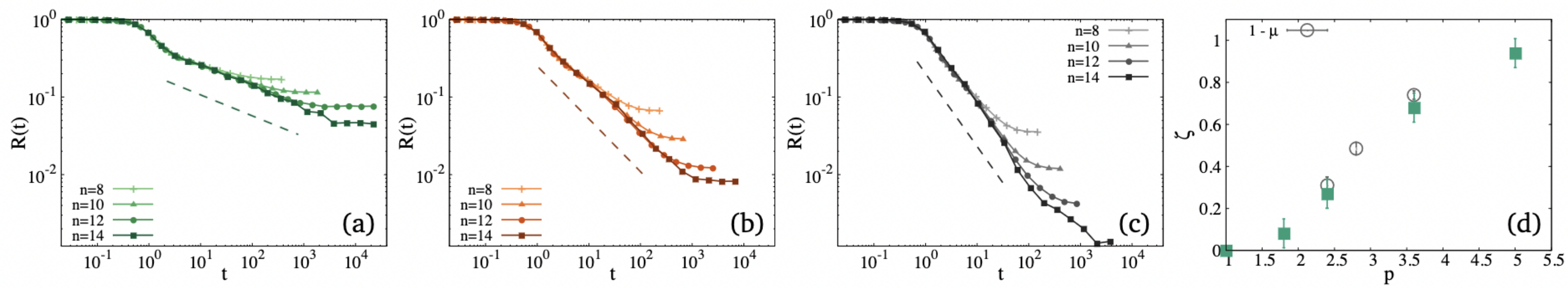


FIG. 12. Return probability, see Eq. (41), as a function of time, for $p = 2.4$ (a), $p = 3.6$ (b), and $p = 5$ (c). The dashed lines show the power-law decay of the return probability as $R(t) \propto t^{-\zeta}$. In panel (d) we plot the exponent ζ upon varying the average degree p, showing that its value is very close to $\zeta \simeq 1 - \mu$ within our numerical accuracy, μ being the exponent that describes the power-law decay of the overlap correlation function $K_2(\omega)$ — see Figs. 10(a–b).

Results

The averaged density of eigenvalues

After some lengthy but simple steps we obtain

$$\rho(\lambda) = -\frac{1}{\pi\zeta} \lim_{\eta\to 0^+} \mathrm{Re}\, C(\lambda - \mathrm{i}\eta)$$

$$C(\lambda) = \mathrm{i}\zeta\, G_a(\lambda + 2\mathrm{i}\, C(\lambda))$$

with G_a the global resolvent of $\mathbb{A}$ and $\zeta = \dfrac{\nu^2}{4N^{\gamma-1}}$

The latter eq. can be solved exactly for some p_a, e.g. Cauchy.

Leading finite size corrections captured

Approximate $\rho_{_N}(\lambda)$ for $\zeta \ll 1$ and any p_a

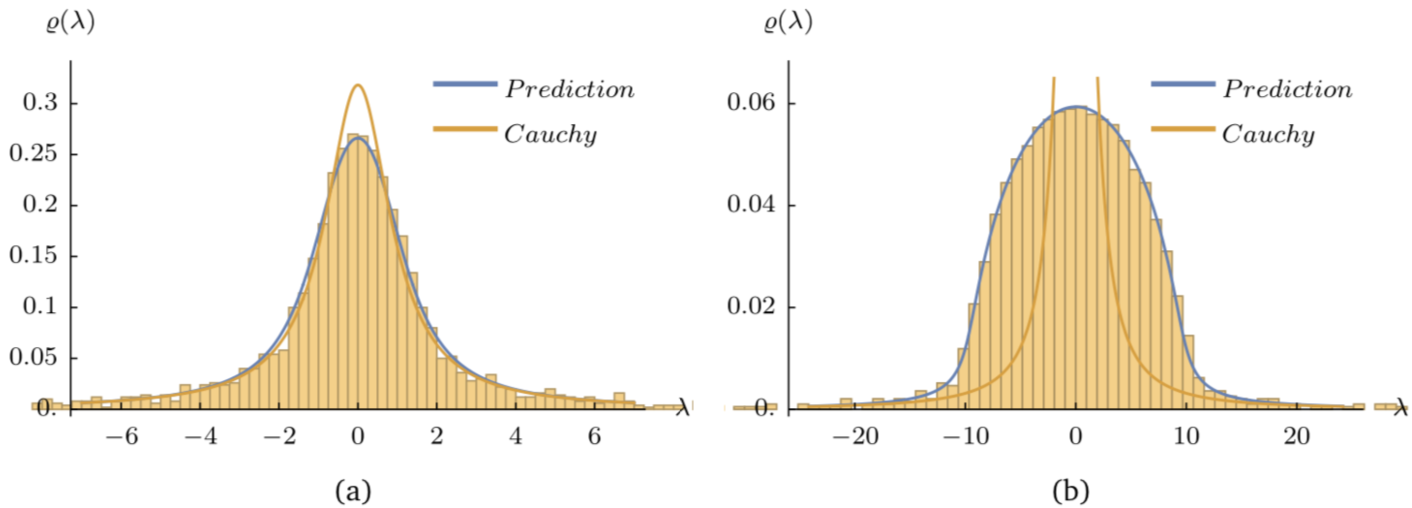
---

Generalization of the **Zee formula** in *Law of addition in random matrix theory*, Nucl. Phys. B 474, 726 (1996)

A. Krajenbrink, P. Le Doussal and N. O'Connell, *Tilted elastic lines with columnar and point disorder, non-Hermitian quantum mechanics, and spiked random matrices: pinning and localization*, Phys. Rev. E 103, 042120 (2021)

Results

The averaged density of eigenvalues



$\zeta = 0.12$
$\nu = 1$

$\zeta = 11.7$
$\nu = 10$

$$\zeta = \frac{\nu^2}{4N^{\gamma-1}}$$

$N = 2000,\ \gamma = 1.1$

Results

The level compressibility

The # of eigenvalues in the interval $[\alpha, \beta]$ is the random variable

$$I(\alpha, \beta) = \int_{\alpha}^{\beta} d\lambda' \, \rho(\lambda')$$

The level compressibility is defined as

$$\chi(\lambda) = \frac{\kappa_2(\lambda)}{\kappa_1(\lambda)} = \frac{[I^2(-\lambda, \lambda)]_{\mathbb{H}} - [I(-\lambda, \lambda)]_{\mathbb{H}}^2}{[I(-\lambda, \lambda)]_{\mathbb{H}}}$$

Limits

– Pure Poisson, typical level spacing $\mathcal{O}(1)$; $\chi(\lambda) \sim 1$ for small λ and $\chi(\lambda) \sim 0$ for large λ

– Pure GOE ($\mathbb{A} = 0$), $\chi(\lambda \gg N^{-1/2-\gamma/2}) \to 0$ and $\chi(\lambda \ll N^{-1/2-\gamma/2}) \to 1$

Metz & Pérez Castillo, *Large Deviation Function for the Number of Eigenvalues of Sparse Random Graphs Inside an Interval*, Phys. Rev. Lett. 117, 104101 (2016)

Metz, *Replica-symmetric approach to the typical eigenvalue fluctuations of Gaussian random matrices*, J. Phys. A 50, 495002 (2017)

Results

The level compressibility

We calculated the cumulant generating function $\boxed{\mathcal{F}_\lambda(s) = \frac{1}{N}\ln[e^{-sI(-\lambda,\lambda)}]_{\mathbb{H}}}$

In the interesting regime $1 < \gamma < 2$ we found

– $\chi(\lambda) \sim 0$ for $\lambda \ll E_T$ within mini-bands (like not-too-small λ GOE)

– $\chi(\lambda) \sim 1$ for $\lambda \gg E_T$ across mini-bands (like small λ Poisson)

– In the scaling limit $y = \dfrac{\lambda}{2\pi p_a(0)\zeta}$ with $\zeta = \dfrac{\nu^2}{4N^{\gamma-1}}$, a universal form

$$\boxed{\overline{\chi}(y) = \frac{1}{\pi y}[2y\,\text{arctan}(y) - \ln(1+y^2)]}$$

with $\overline{\chi}(y \to 0) = 0$ and $\overline{\chi}(y \to \infty) = 1$

Numerical tests of universality (independence of p_a) are under way

Details

On level compressibility

The # of eigenvalues in the interval $[\alpha, \beta]$ is

$$I(\alpha, \beta) = \sum_{i=1}^{N} [\theta(\beta - \lambda_i) - \theta(\alpha - \lambda_i)]$$

The Heaviside function can be represented as

$$\theta(-x) = \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} [\ln(x + \mathrm{i}\eta) - \ln(x - \mathrm{i}\eta)]$$

Then

$$\sum_{i=1}^{N} \theta(\alpha - \lambda_i) = \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} \{\ln \det[\mathbb{H} - (\alpha - \mathrm{i}\eta)\mathbb{I}] - \ln \det[\mathbb{H} - (\alpha + \mathrm{i}\eta)\mathbb{I}]\}$$

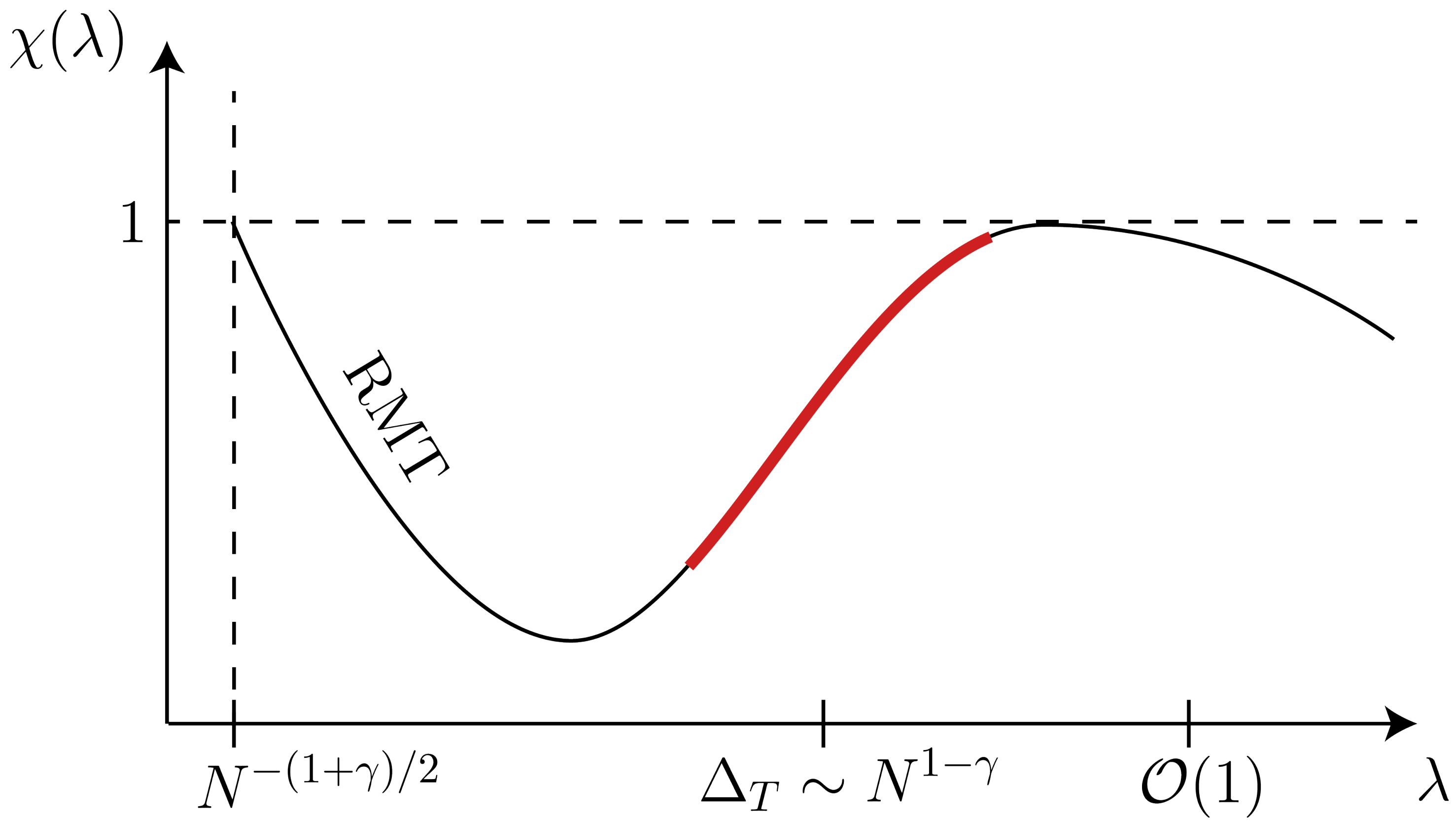
and

$$I(-\alpha, \beta) = -\frac{1}{\pi \mathrm{i}} \lim_{\eta \to 0^+} \ln \frac{\mathcal{Z}(\beta - \mathrm{i}\eta)\, \mathcal{Z}(\alpha + \mathrm{i}\eta)}{\mathcal{Z}(\beta + \mathrm{i}\eta) \mathcal{Z}(\alpha - \mathrm{i}\eta)}$$

+ replica trick

Level compressibility

Sketch of the various scales



$\overline{\chi}(y)$

Methods

The trace of the **resolvent matrix** $\mathbb{G}(z) = (z\mathbb{I} - \mathbb{H})^{-1}$ is the **global resolvent**

$$G(z) \equiv \tfrac{1}{N} \mathrm{Tr}\, \mathbb{G}(z) = \tfrac{1}{N} \sum_{i=1}^{N} (z - \lambda_i)^{-1} \xrightarrow[N\to\infty]{} \int d\lambda' \, \frac{\rho(\lambda')}{z - \lambda'}$$

Inverting

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \lim_{N\to\infty} G(\lambda - \mathrm{i}\eta)$$

$$= \frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \lim_{N\to\infty} \tfrac{1}{N} \frac{\partial}{\partial\lambda} \sum_{i=1}^{N} \ln(\lambda - \mathrm{i}\eta - \lambda_i)$$

With the **Edwards-Jones** Gaussian representation

$$[\rho(\lambda)]_{\mathbb{H}} = -\frac{1}{\pi} \lim_{\eta\to 0^+} \mathrm{Im} \frac{\partial}{\partial\lambda} [\ln \mathcal{Z}(\lambda - \mathrm{i}\eta)]_{\mathbb{H}}$$

$$\mathcal{Z}(z) = \frac{1}{(2\pi\mathrm{i})^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N r \, e^{-\frac{1}{2}\mathbf{r}^{\mathrm{T}} (z\mathbb{I} - \mathbb{H})\, \mathbf{r}}$$

Edwards & Jones, *The eigenvalue spectrum of a large symmetric random matrix*, J. Phys. A 9, 1595 (1976)

Methods

The replica trick

$$[\ln \mathcal{Z}]_{\mathbb{H}} = \lim_{n\to 0} \frac{[\mathcal{Z}^n]_{\mathbb{H}} - 1}{n} = \lim_{n\to 0} \frac{1}{n} \ln [\mathcal{Z}^n]_{\mathbb{H}}$$

For $N \to \infty$ the calculation reduces to the saddle-point evaluation of $[\mathcal{Z}^n]_{\mathbb{H}}$

– it can be done with a replica symmetric Ansatz on $NQ_{ab} = \langle \mathbf{r}^a \cdot \mathbf{r}^b \rangle$ as usual

– with a **rotationally invariant Ansatz** in replica space for the density

$$\mu(\vec{r}) = \mu(r^1, \dots, r^a) = \frac{1}{N} \sum_{i=1}^{N} \prod_{a=1}^{n} \delta(r^a - r_i^a)$$

such that at the saddle point level it only depends on the modulus $\mu(\vec{r}) = \overline{\mu}(r)$

The second path turns out to be more convenient

Livan, Novaes & Vivo, *Introduction to Random Matrices – Theory and Practice*, arXiv : 1712.07903, SpringerBriefs in Mathematical Physics 26 (2018)