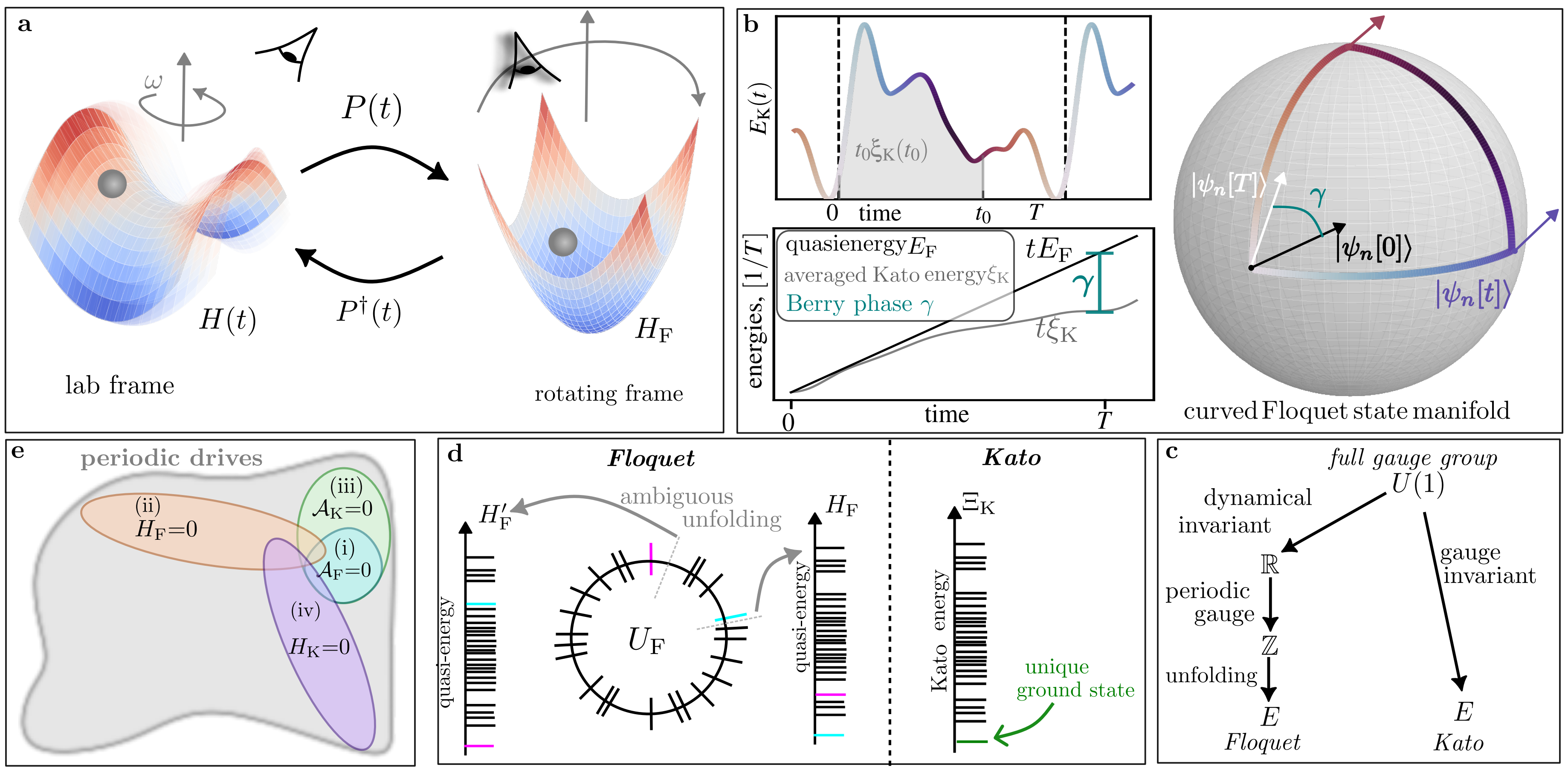
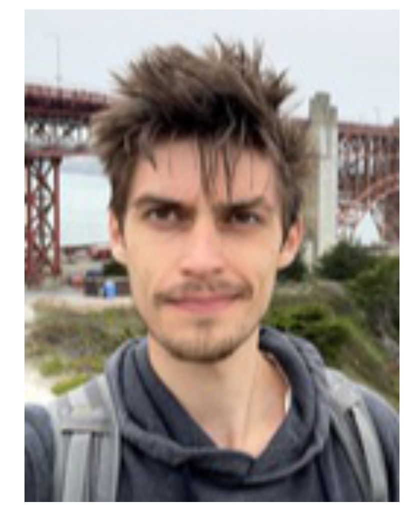
MAX PLANCK INSTITUTE
FOR THE PHYSICS OF COMPLEX SYSTEMS

mpipks

Geometric Floquet Theory





Paul M Schindler

MAX PLANCK GESELLSCHAFT

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Deutsche Forschungsgemeinschaft

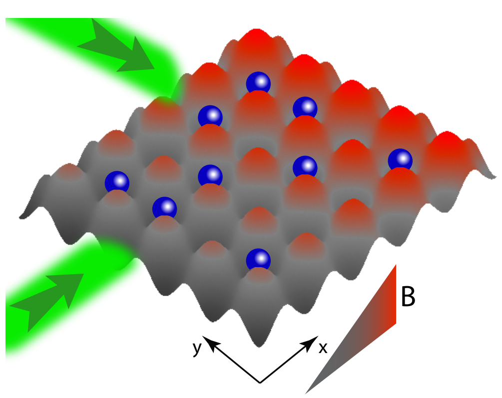
PM Schindler and MB, arXiv: 2410.07029

Periodically driven systems

- why care about periodic drives in quantum systems?

quantum simulation

- Floquet engineering
  - artificial gauge fields
  - dynamical localization
  - topological matter
- nonequilibrium ordered states
  - time crystals, etc.



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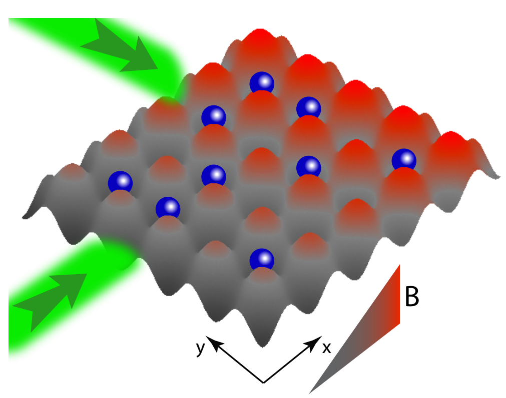
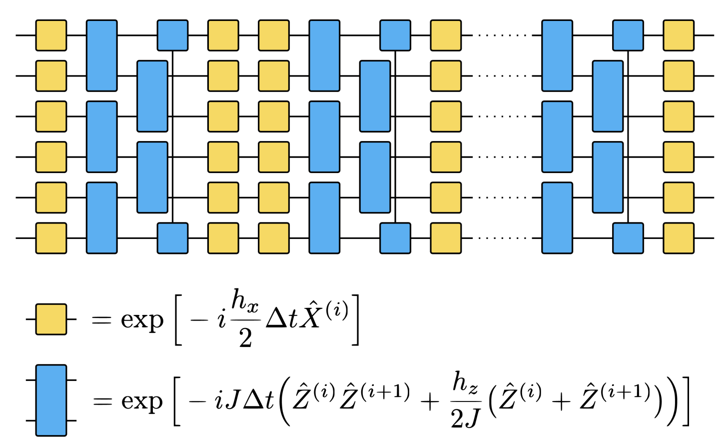


quantum computing

- quantum algorithms
  - Trotterization
- Floquet unitary circuits
- error correction
  - Floquet codes



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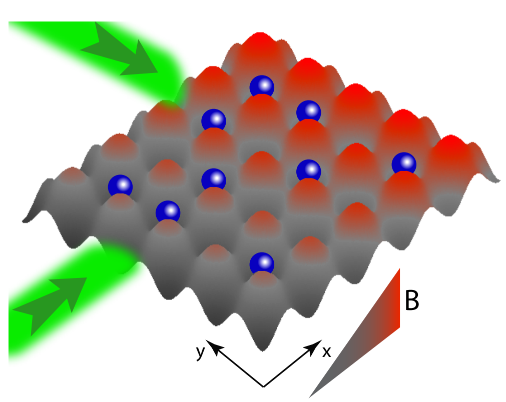

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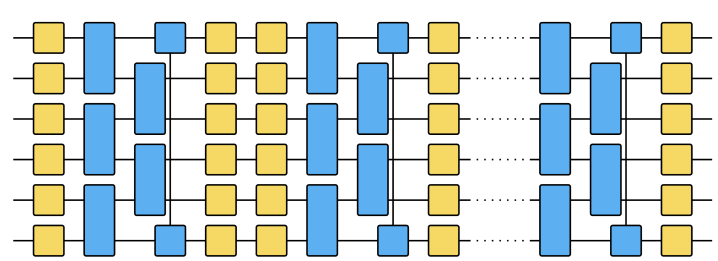

$$= \exp\left[-i\frac{h_x}{2}\Delta t \hat{X}^{(i)}\right]$$

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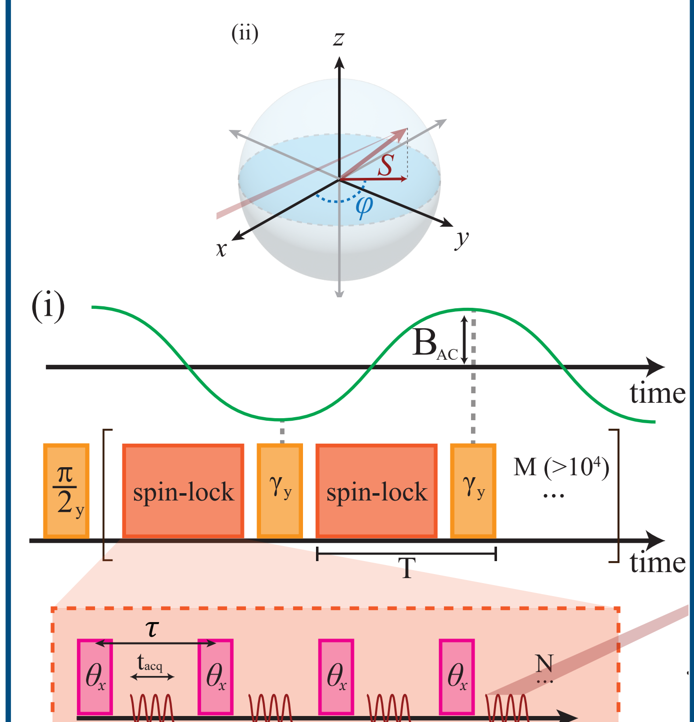
quantum sensing

- dynamical decoupling
- Ramsey interferometry



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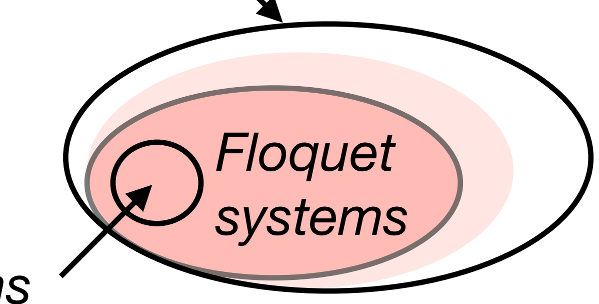

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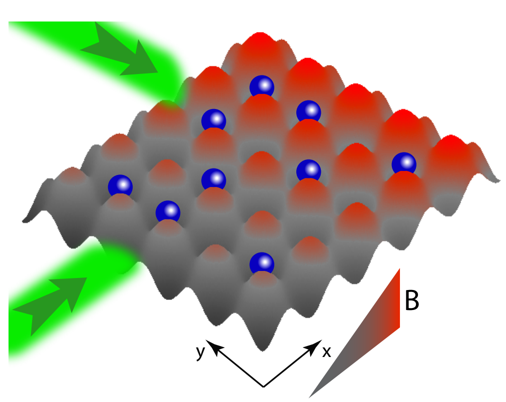

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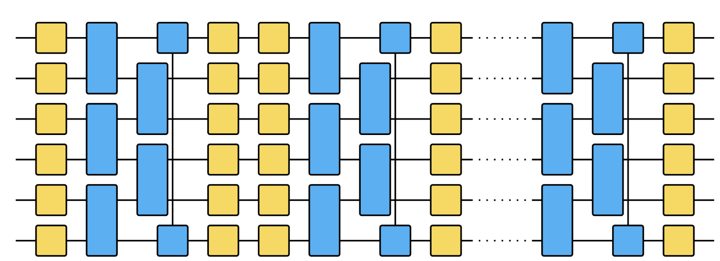

[yellow gate] $= \exp\left[-i\frac{h_x}{2}\Delta t \hat{X}^{(i)}\right]$

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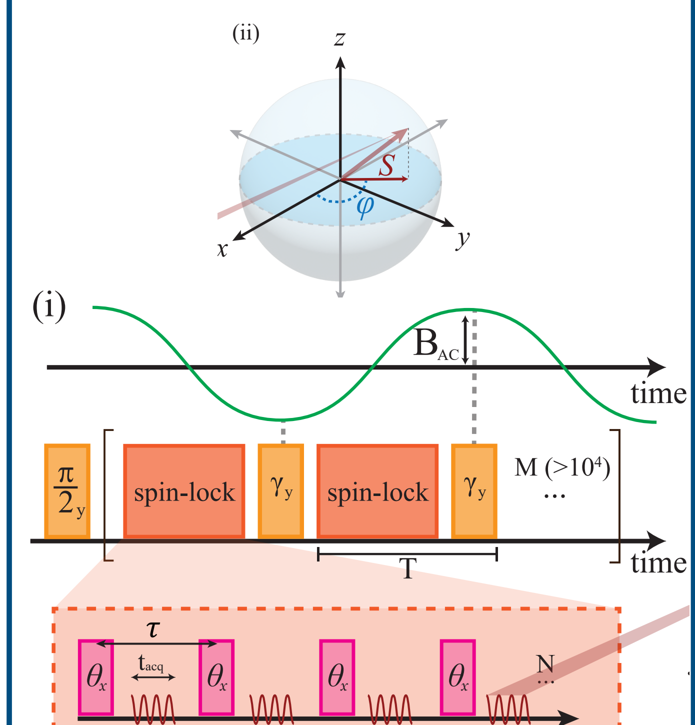
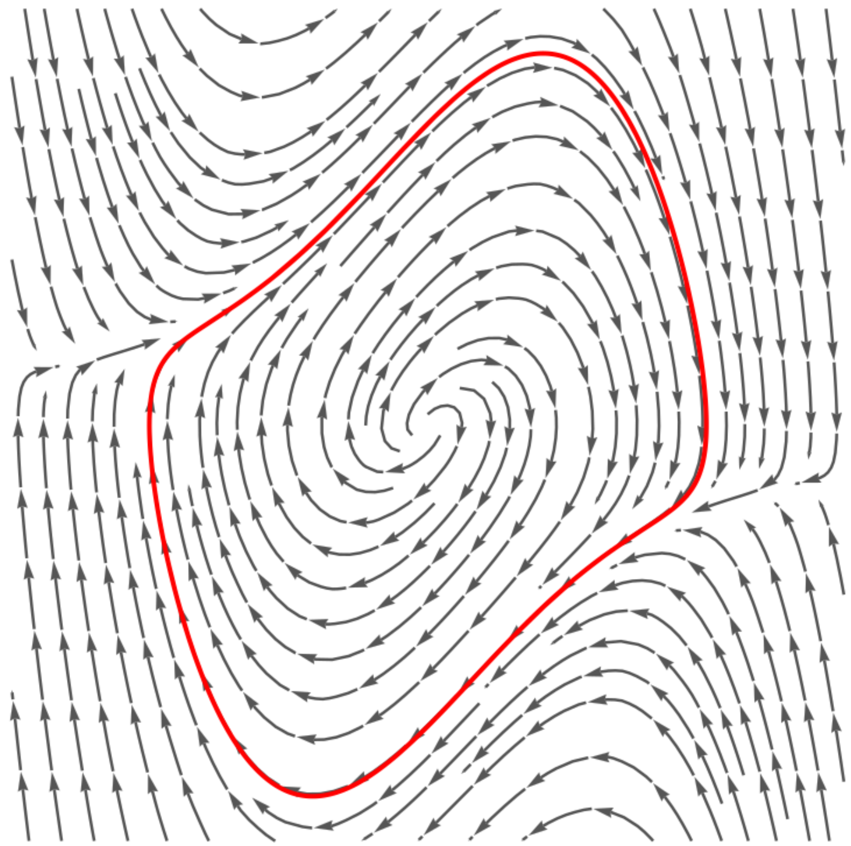


Q: how do we manipulate periodically driven systems?

Floquet theory

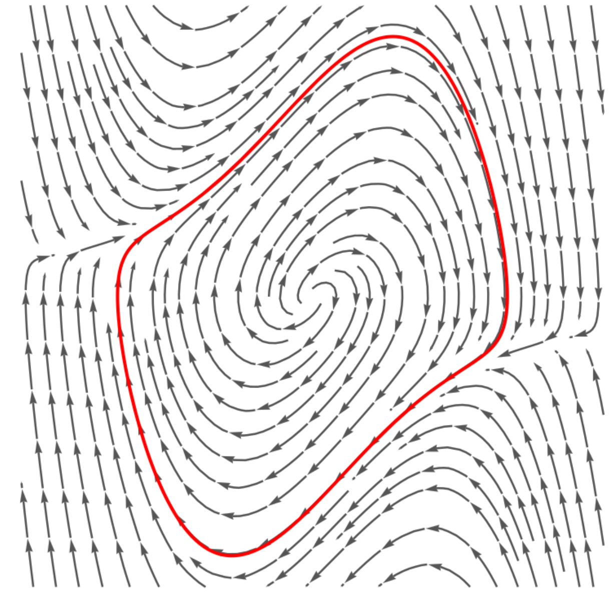
- **Floquet (1883)** $\dot{\psi}(t) = -iH(t)\psi(t),$ **linear &** $H(t+T) = H(t)$

phase space



van der Pol oscillator

Floquet theory



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micromotion
$P(t) = P(t+T)$

Floquet Hamiltonian,
time-**<u>in</u>**dependent

effective object (!)
does not exist w/o drive

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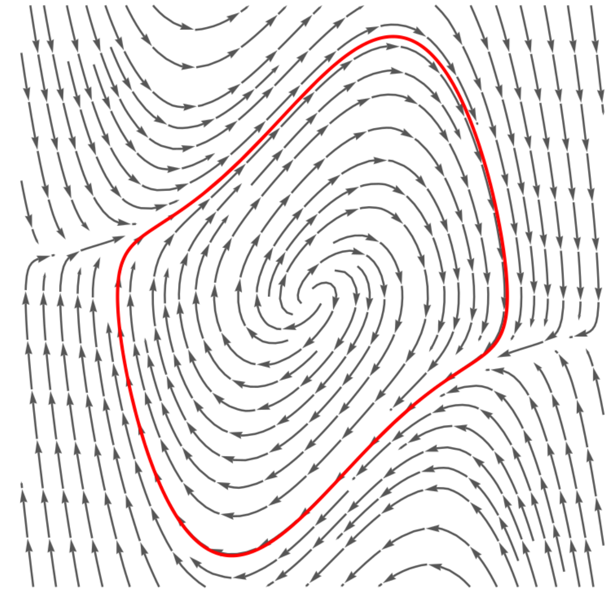

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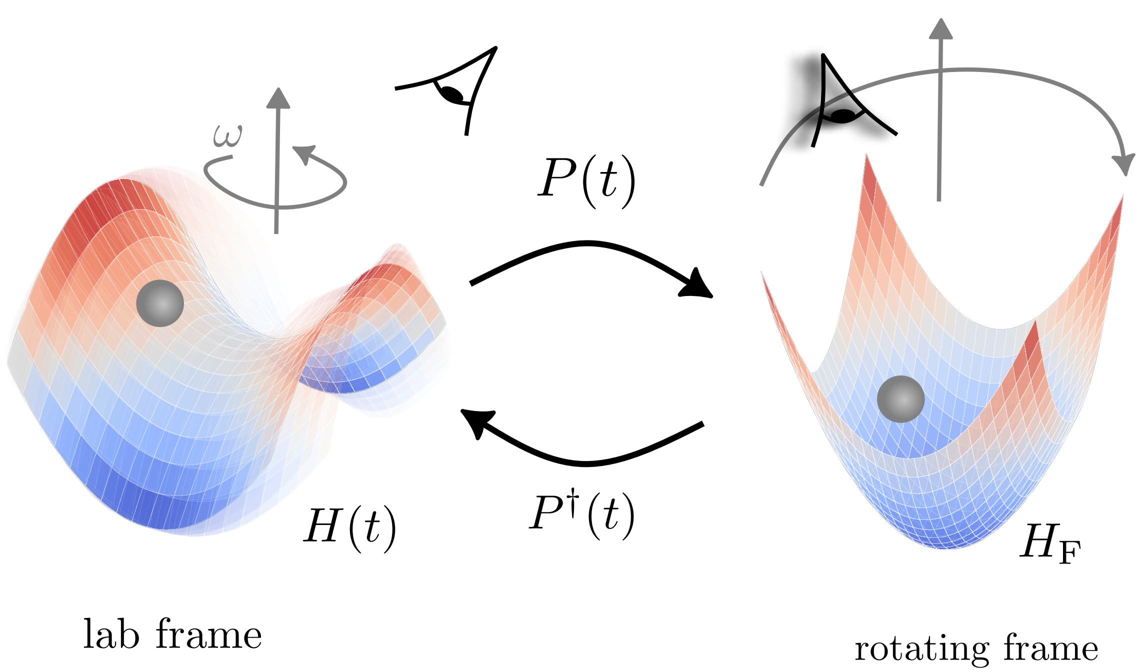
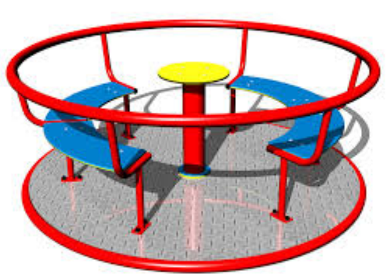
physical meaning: $H_F[0] = P^\dagger(t)H(t)P(t) - P^\dagger(t)i\partial_t P(t)$

distinct rotating frame

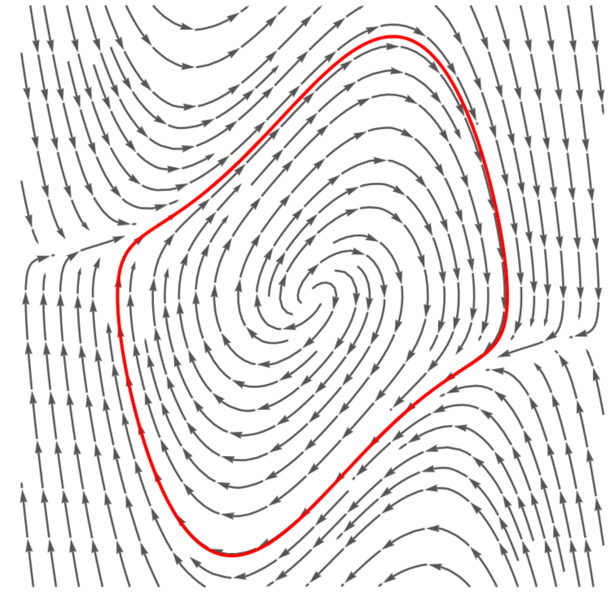


Merry-go-round



experts

Floquet theory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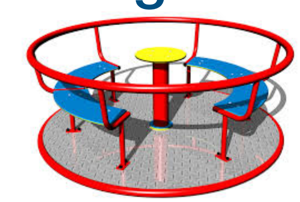


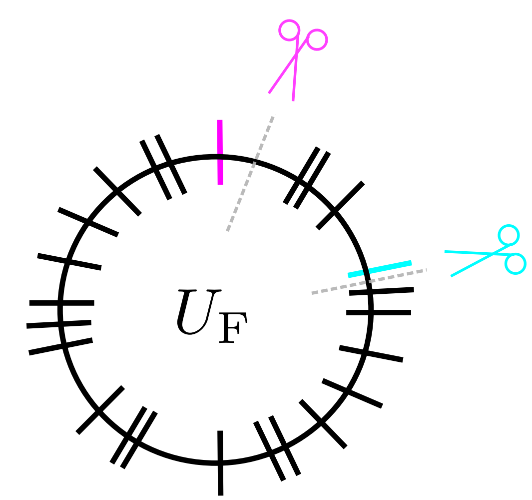
- **Floquet eigenvalue problem** $H_F|n_F\rangle = \varepsilon_F^{(n)}|n_F\rangle$

Floquet states

quasi-energy: defined up to $\varepsilon_F \rightarrow \varepsilon_F + m\omega$

H_F not unique (!)

issue: quasi-energy not ordered (“no Floquet ground state”)



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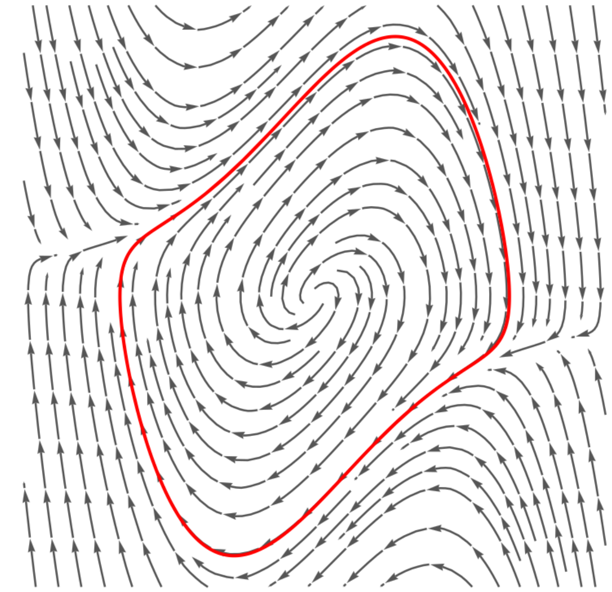

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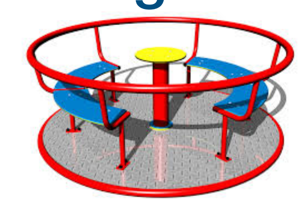

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phase of drive: $H(t-t_0)$

$H_F[t_0] = P(t_0)H_F[0]P^\dagger(t_0)$ $\quad |n_F[t_0]\rangle = P(t_0)|n_F[0]\rangle$

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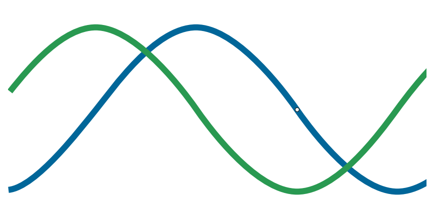

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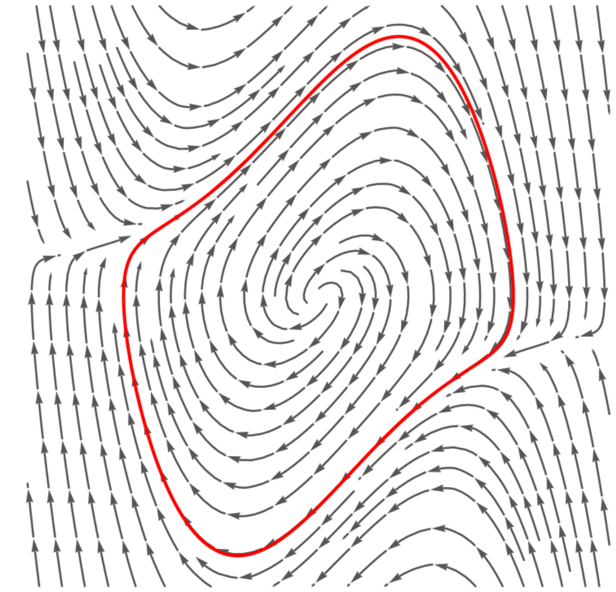

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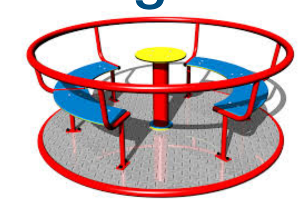

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$$P(t)|n_F[0]\rangle = |n_F[t]\rangle \neq |n_F(t)\rangle = P(t)e^{-itH_F}|n_F[0]\rangle = e^{-it\varepsilon_F^{(n)}}|n_F[t]\rangle$$

instantaneous $\neq$ evolved

How do we find H_F ?

- **solve Schrödinger equation**
  - exact solutions: limited (circular drives, harmonic oscillators, etc.)
  - in general: compute time-ordered exponentials → special functions

Floquet theorem

$$U(T,0) = \mathcal{T} \exp\left(-i\int_0^T \mathrm{d}t H(t)\right) = \exp(-iTH_F)$$

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origin: H_F **is non-local:** $H_F = \sum_n \varepsilon_F^{(n)} |n_F\rangle\langle n_F| \longrightarrow \sum_n \varepsilon_F^{(n)} |n_F\rangle\langle n_F| + \omega|m_F\rangle\langle m_F| = H_F'$

e'state projector

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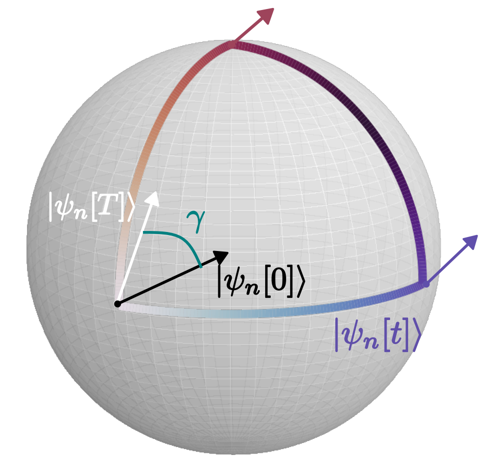
Q: other approaches to describe Floquet systems?

Geometric Floquet theory
(take-home messages)

- **Floquet theory follows from the adiabatic theorem**
  - alternative decomposition of dynamics: geometric & dynamical phases



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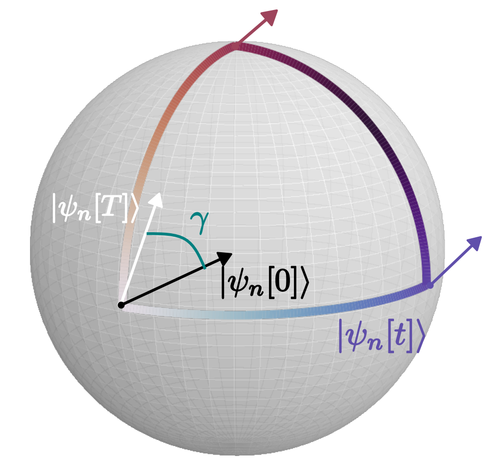

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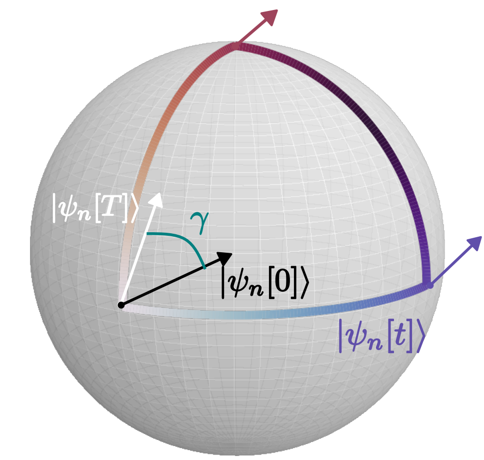

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  - alternative decomposition of dynamics: geometric & dynamical phases
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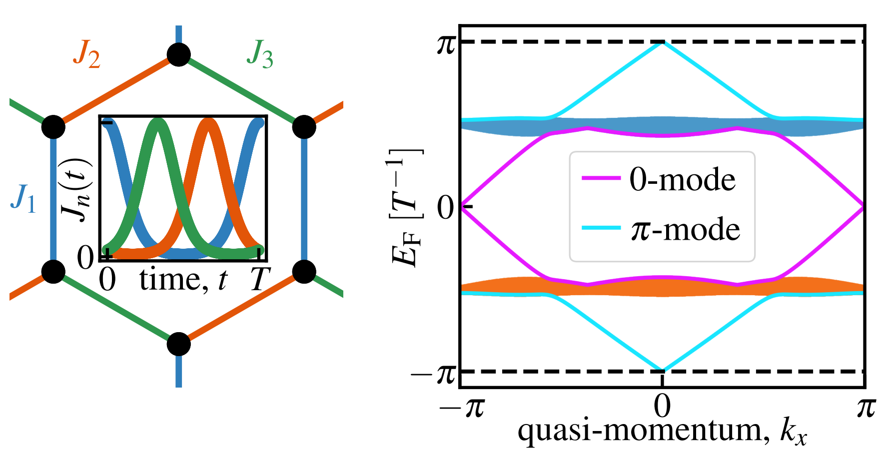
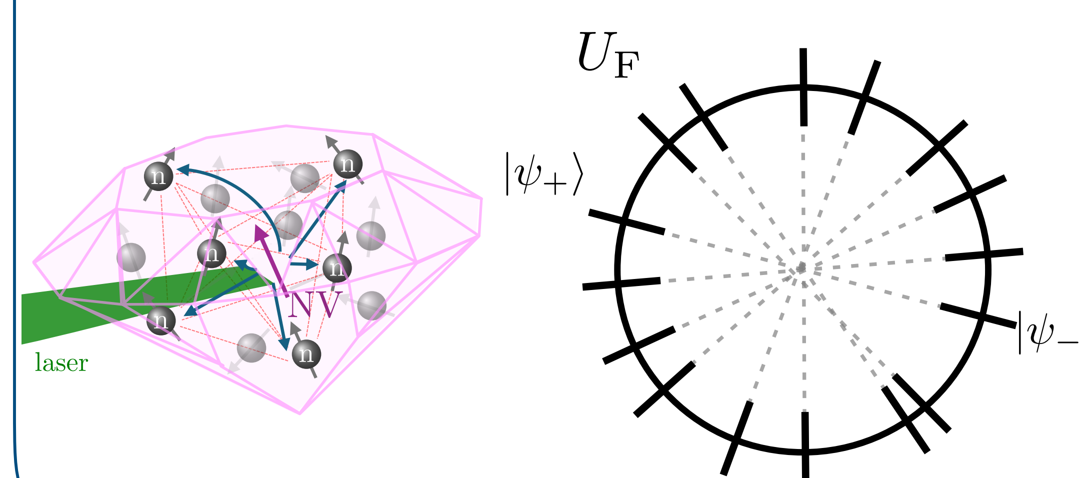
  - guaranteed by parallel-transport gauge and the adiabatic limit
- **geometric phase captures inherently nonequilibrium phenomena**

π-modes in Floquet anomalous topological insulators



discrete time crystals



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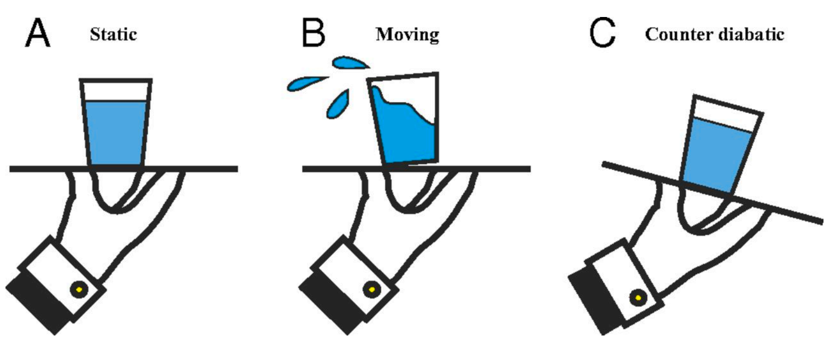
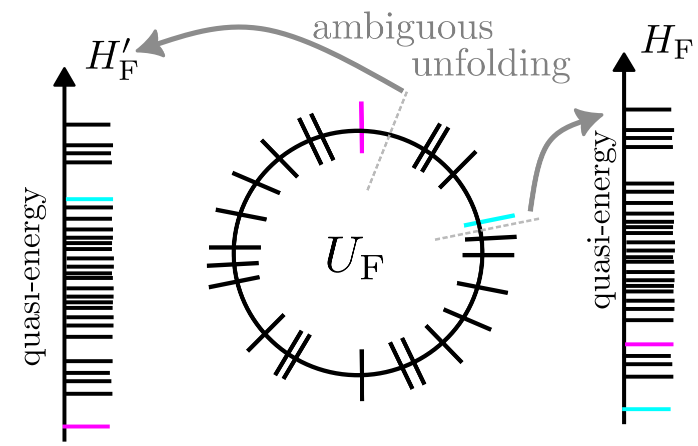
Outline

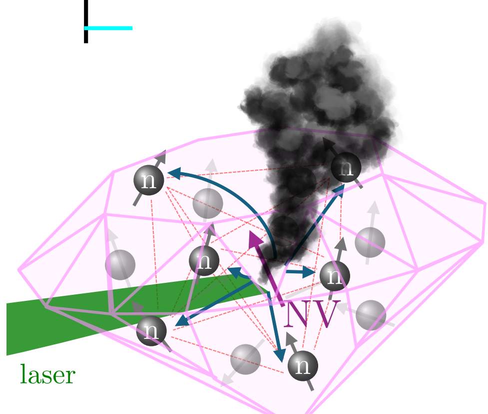
- **Adiabatic evolution**
  - adiabatic gauge potentials
  - counterdiabatic driving



- **Geometric Floquet theory**
  - Floquet theory as a shortcut to adiabaticity
  - quasienergy folding
  - *the* Floquet ground state



- **Applications**
  - heating, discrete time crystals
  - variational principle for Floquet Hamiltonian



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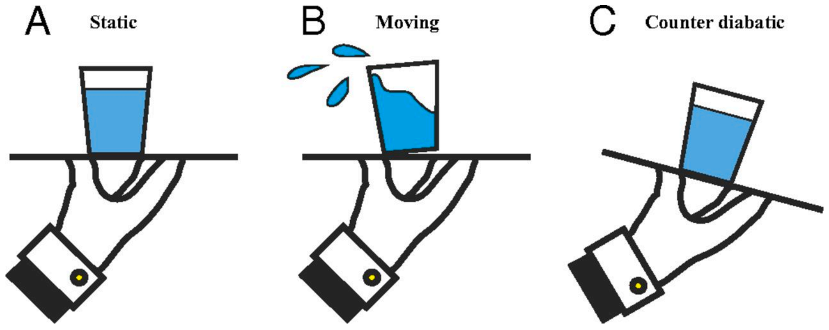

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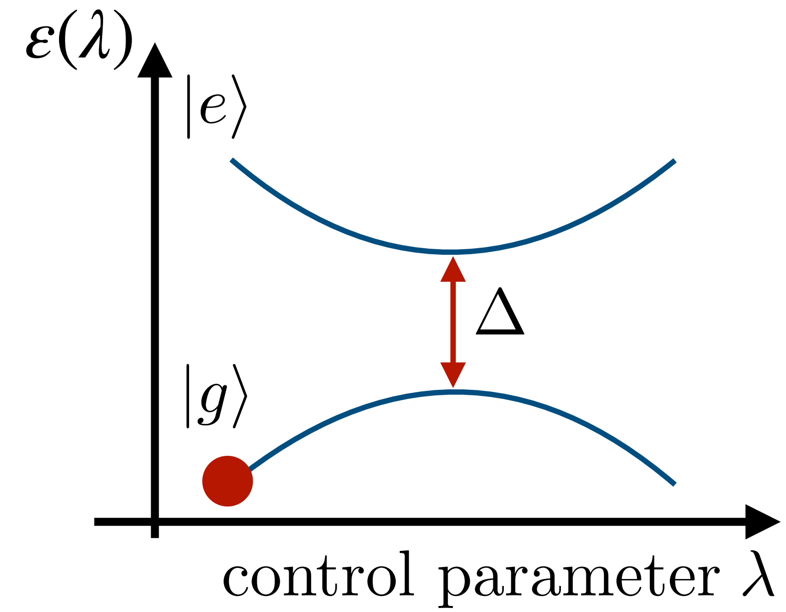
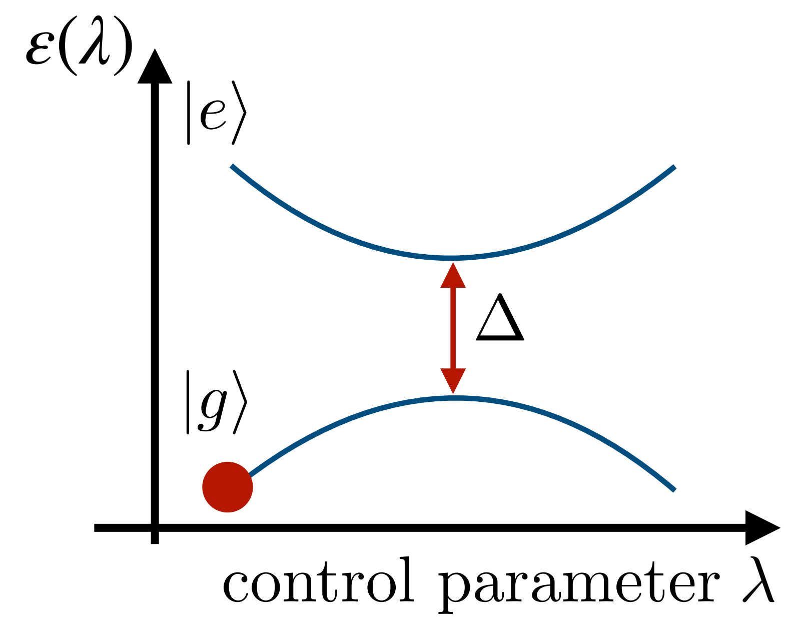
Adiabatic driving



Landau Zener problem

- **adiabatic theorem**
  - gapped e'state $H(\lambda)|n[\lambda]\rangle = \varepsilon(\lambda)|n[\lambda]\rangle$
  - adiabatic limit: $\dot{\lambda} \to 0, \quad T \to \infty, \quad \dot{\lambda}T \to$ const.

Adiabatic driving



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$$|n(t)\rangle = \mathcal{T}\exp\left(-i\int_0^t \mathrm{d}s H(\lambda(s))\right)|n[0]\rangle \to e^{-i\phi_n(t)}e^{-i\gamma_n(t)}|n[\lambda(t)]\rangle$$

evolved state — instantaneous state

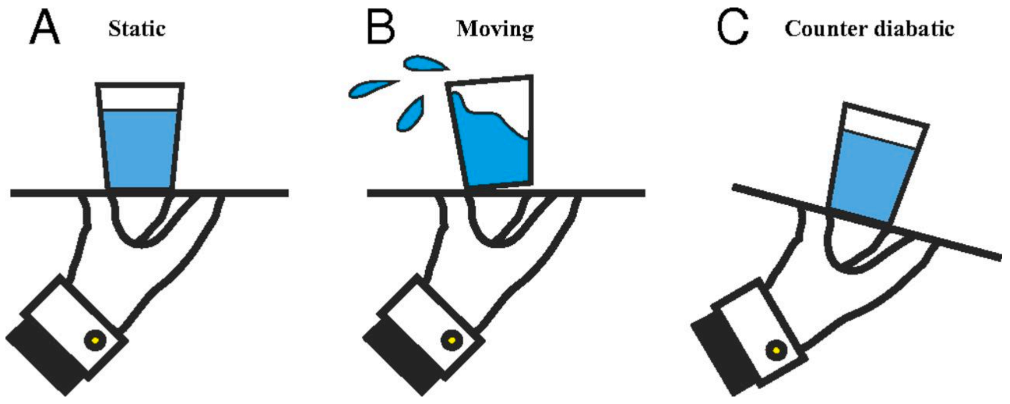
✦ **dynamical phase** $\phi_n(t) = \int_0^t \mathrm{d}s\ \varepsilon(\lambda(s))$

✦ **geometric phase** $\gamma_n(t) = \int_{\lambda(0)}^{\lambda(t)} \mathrm{d}\lambda\ \langle n[\lambda]|i\partial_\lambda|n[\lambda]\rangle$

path-independent

Counterdiabatic driving

- **breakdown of adiabatic evolution away from adiabatic limit**
  - get rid of excitations by applying a counter-force
  - shortcut to adiabaticity



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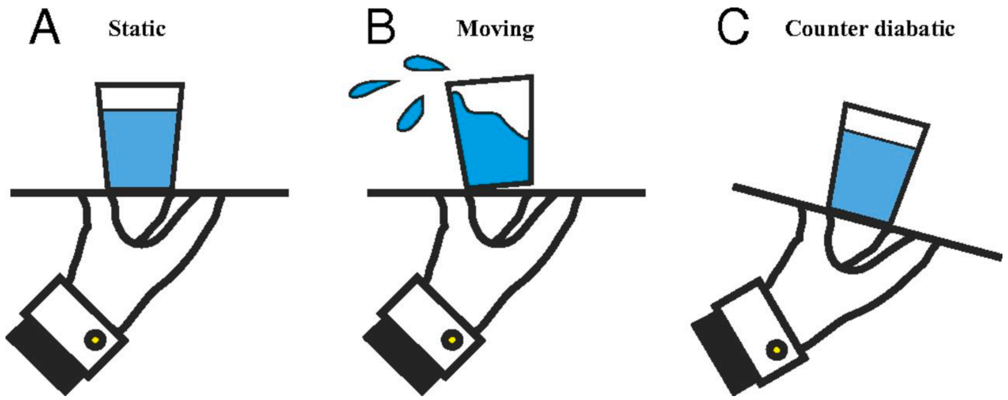

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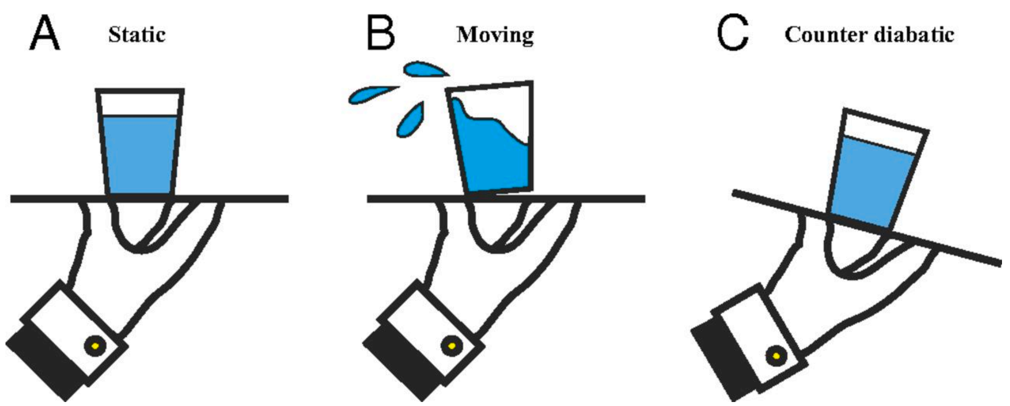

- counterdiabatic (CD) driving $H_{\rm CD}(\lambda) = H(\lambda) + \dot\lambda \mathscr{A}_\lambda$
- identify cause of excitations
  - diagonalizing unitary: $U^\dagger(\lambda) H(\lambda) U(\lambda) = D_\lambda$
  - co-moving frame Hamiltonian: $H_{\rm co-mov} = U^\dagger H U - \dot\lambda U^\dagger i\partial_\lambda U = D_\lambda - \dot\lambda \tilde{\mathscr{A}}_\lambda$

diagonal, no excitations

adiabatic gauge potential (AGP) creates all excitations

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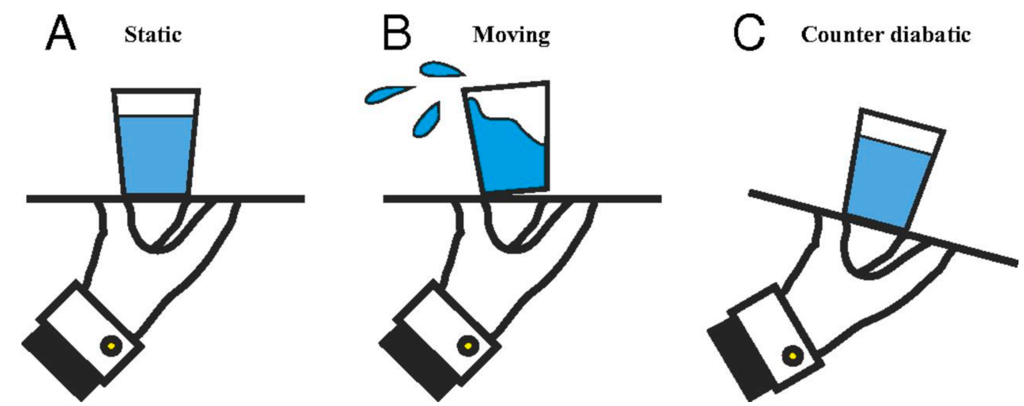

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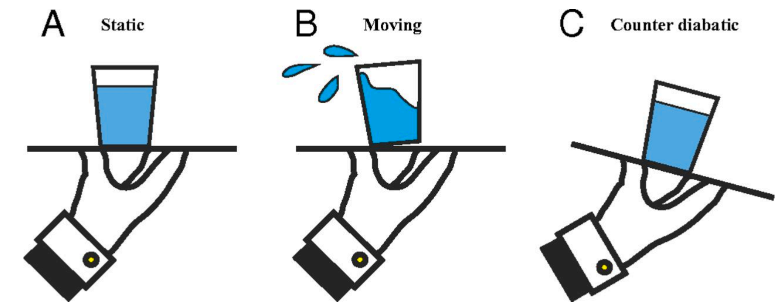
diagonal, no excitations → D_λ

adiabatic gauge potential (AGP) creates all excitations → $\dot{\lambda}\tilde{\mathscr{A}}_\lambda$

- Berry connection: $A_\lambda^{(n)} = \langle n[\lambda] | \mathscr{A} | n[\lambda] \rangle$
- Berry phase: $\gamma = \oint_C A_\lambda^{(n)} \cdot \mathrm{d}\lambda$
- geometric tensor: $g_{\mu\nu}^{(n)} = \langle n[\lambda] | \mathscr{A}_\mu \mathscr{A}_\nu | n[\lambda] \rangle_c$
- Berry curvature: $F_{\mu\nu}^{(n)} = \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$

Gauge potential



- **AGP not unique: U(1) gauge freedom**

Berry connection *not* gauge invariant!

  - re-phase e'state: $|n[\lambda]\rangle \mapsto e^{i\chi_n(\lambda)}|n[\lambda]\rangle$

$\langle n|\mathscr{A}_\lambda|n\rangle \to \langle n|\mathscr{A}_\lambda|n\rangle - \partial_\lambda\chi_n$

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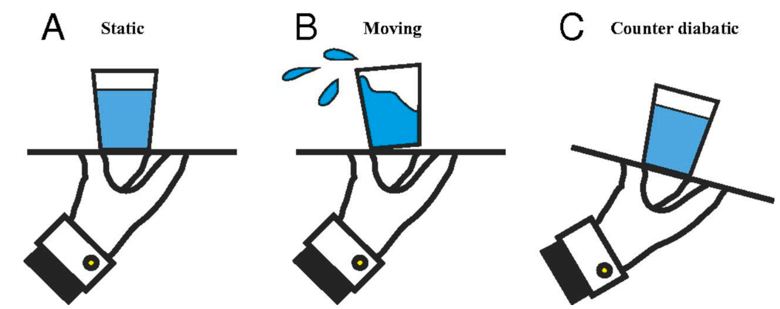
experts

Gauge potential



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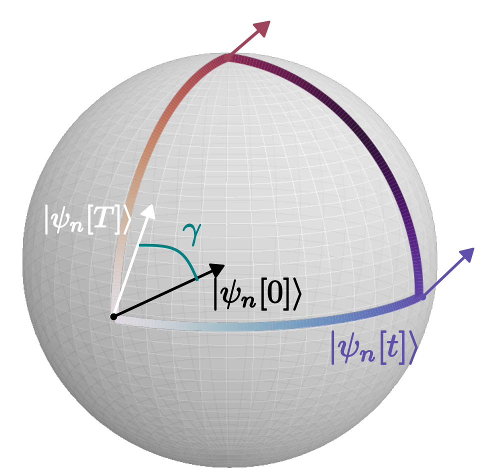


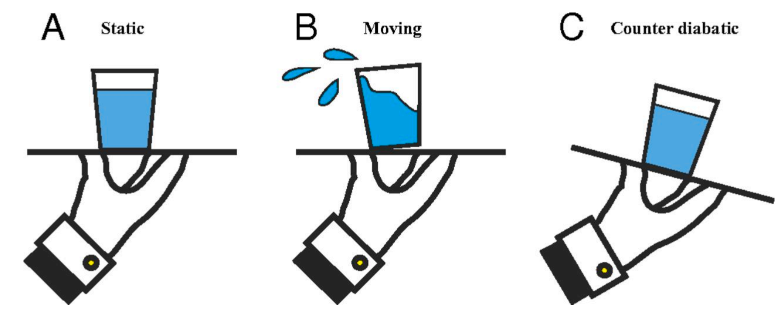
$\mathscr{A}_\lambda \leftrightarrow i\partial_\lambda$ derivative

$\mathscr{A}_K \leftrightarrow iD_\lambda$

covariant derivative

experts

Gauge potential



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  - unique: CD driving reproduces adiabatic phases

adiabatic limit

$$|n(t)\rangle = \mathscr{T}\exp\left(-i\int_0^t \mathrm{d}s\, H(\lambda(s))\right)|n(0)\rangle \to e^{i\phi_n(t)}e^{i\gamma_n(t)}|n[\lambda(t)]\rangle$$

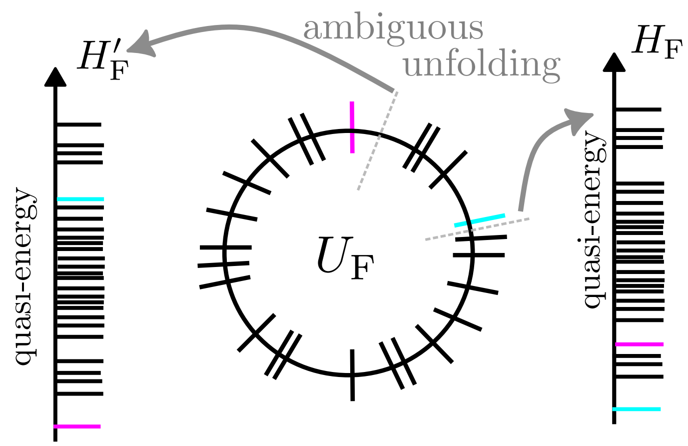
evolved state $\approx$ / $=$ $e^{i\,\mathrm{phase}}$ instantaneous state

CD driving

$$|n(t)\rangle = \mathscr{T}\exp\left(-i\int_0^t \mathrm{d}s\, \overbrace{H(\lambda(s)) + \dot\lambda\mathscr{A}_K(\lambda(s))}^{=H_{\mathrm{CD}}}\right)|n(0)\rangle = e^{i\phi_n(t)}e^{i\gamma_n(t)}|n[\lambda(t)]\rangle$$

Outline

- **Geometric Floquet theory**
  - Floquet theory as a shortcut to adiabaticity
  - quasienergy folding
  - *the* Floquet ground state



- **Applications**
  - heating, discrete time crystals
  - variational principle for Floquet Hamiltonian

PM Schindler and MB, arXiv: 2410.07029

Floquet theory as a shortcut to adiabaticity

- **Floquet's theorem:**

$$H_F[0] = P^\dagger(t)H(t)P(t) - P^\dagger(t)i\partial_t P(t)$$

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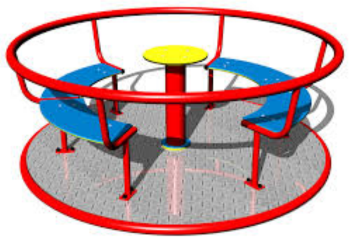

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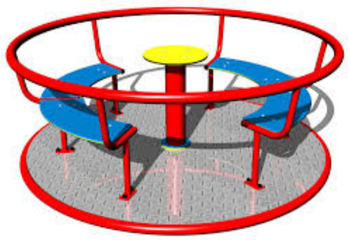

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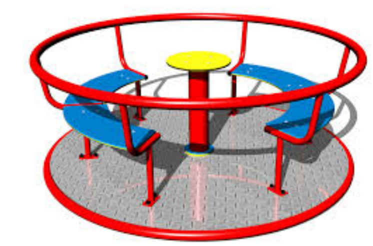

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- check: $|n_F(t)\rangle = \mathcal{T}e^{-i\int_0^t \mathrm{d}s H(s)}|n_F(0)\rangle = P(t)e^{-itH_F}|n_F(0)\rangle = e^{-it\varepsilon_F^{(n)}}P(t)|n_F[0]\rangle = e^{-it\varepsilon_F^{(n)}}|n_F[t]\rangle$

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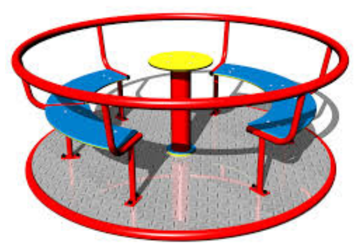

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- **given drive $H(t)$, finding AGP $\mathscr{A}_F(t)$ determines Floquet Hamiltonian $H_F[t]$**
  - variational principle for $\mathscr{A}_F(t)$ gives nonperturbative approximation to $H_F[t]$

❖ Floquet's theorem: special case of the Adiabatic theorem

- adiabatic theorem (in counterdiabatic form) for $\lambda \hat{=} t$:
  - $H_{\mathrm{CD}} = H(t) = H_F[t] + \mathscr{A}_F(t)$ generates adiabatic evolution w.r.t. the states of $H_F[t]$

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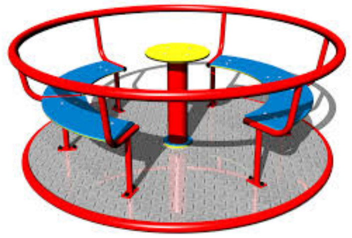
  - co-moving frame: $\tilde{H}(t) = \tilde{H}_F[t]$, i.e., no excitations: $\tilde{U}(t,0) = \exp(-it\tilde{H}_F[0])$

    Floquet rotating frame is the co-moving frame for H_F w.r.t. time



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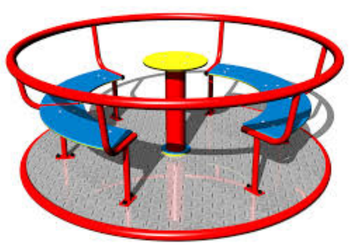


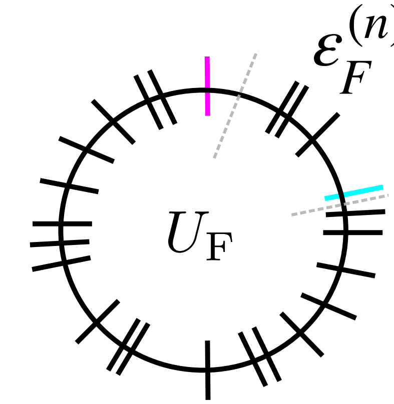
  - evolution in lab frame: $U(t,0) = \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_F(s)\mathrm{d}s\right)\ \exp(-itH_F[0])$

$$= P(t)\ \exp(-itH_F[0])$$

recover Floquet's theorem

for general proof: **PM Schindler and MB, arXiv: 2410.07029**

Quasienergy folding: a new perspective



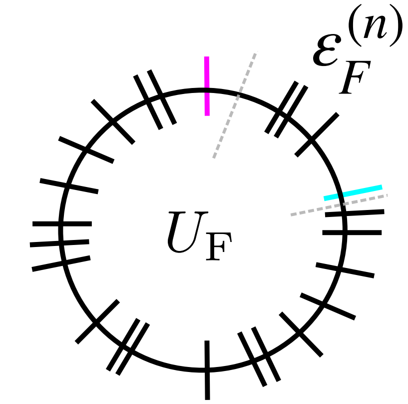
quasienergy spectrum

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

$$U_F = \exp(-iTH_F)$$

- recall: quasienergies defined up to integer multiple of drive frequency: $\varepsilon_F^{(n)} + m\omega, \ m \in \mathbb{Z}$

Quasienergy folding: a new perspective



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$$H(t) = H_F[t] + \mathscr{A}_F(t) \;\mapsto\; H_F[t] - \sum_n \partial_t \chi_n(t) \,|\, n_F[t]\rangle\langle n_F[t] \,| + \mathscr{A}_F(t)$$

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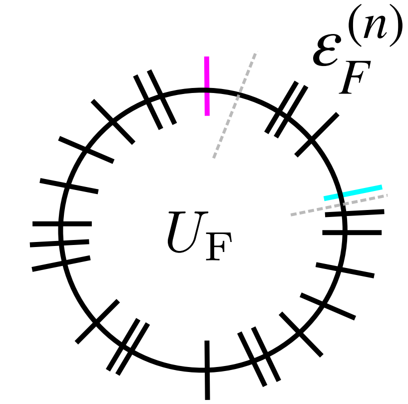

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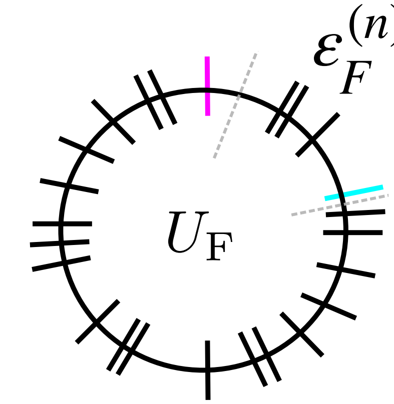

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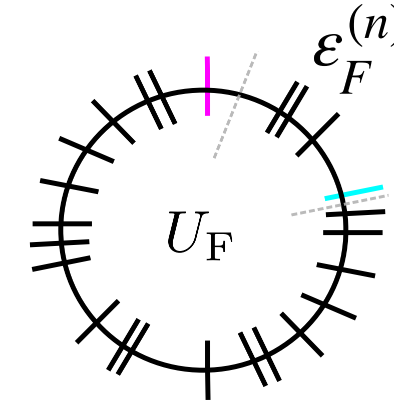

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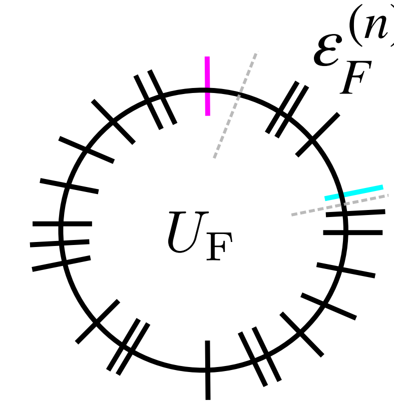
Quasienergy folding: a new perspective



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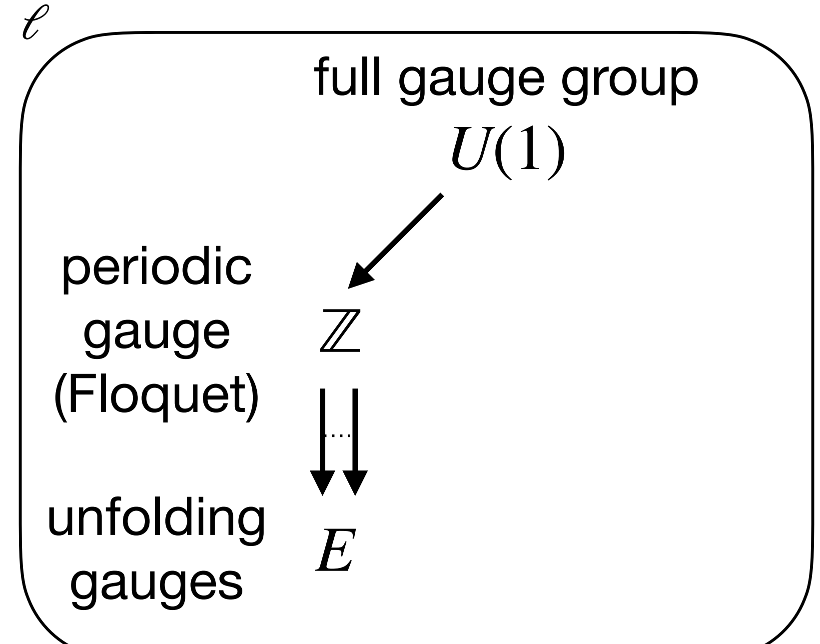
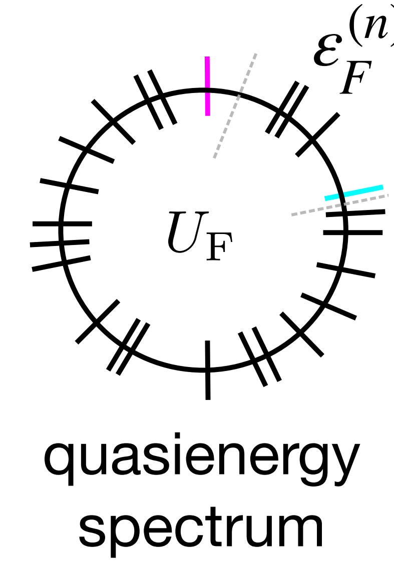


Quasienergy folding: a new perspective



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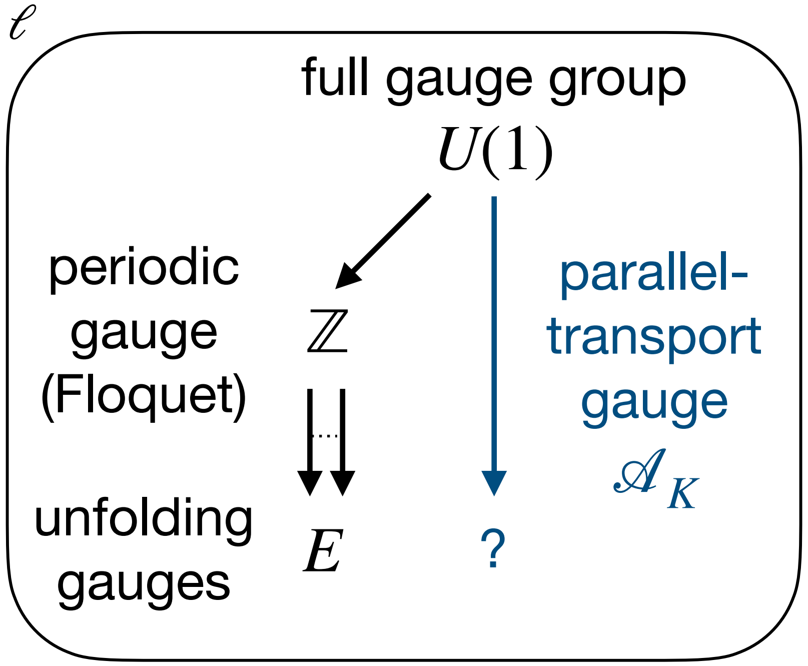

quasienergy folding is a consequence of partial gauge fixing

❖ *The* Floquet ground state

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

micromotion quasienergy

- **evolution operator** $U(t,0) = \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_F(s)\mathrm{d}s\right)\exp(-itH_F[0])$ periodic gauge
  - use Kato potential $\mathscr{A}_K$ $= \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_K(s)\mathrm{d}s\right)\exp(-it\,\text{Æ}(t,0))$ parallel-transport gauge

geometric phase dynamical phase

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geometric phase dynamical phase

  - **A**verage **E**nergy operator $Æ$ and H_F share same e'states (Floquet states)

$$Æ(t,0) = \sum_n æ_n(t,0)\,|n_F[0]\rangle\langle n_F[0]| \qquad æ_n(t,0) = \frac{1}{t}\int_0^t \mathrm{d}s\,\langle n_F[s]\,|\,H(s)\,|\,n_F[s]\rangle$$

unfolded since $H(t)$ is extensive

★ *order Floquet states*

❖ *The* Floquet ground state

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

micromotion quasienergy

- **evolution operator** $U(t,0) = \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_F(s)\mathrm{d}s\right)\exp(-itH_F[0])$ periodic gauge

  - use Kato potential $\mathscr{A}_K$ $= \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_K(s)\mathrm{d}s\right)\exp(-it\,Æ(t,0))$ parallel-transport gauge

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Wilson loop, Berry phases

period-averaged energy
indep. of phase of the drive

$$\varepsilon_F^{(n)} = T^{-1}\gamma_n(T) + æ_n(T)$$

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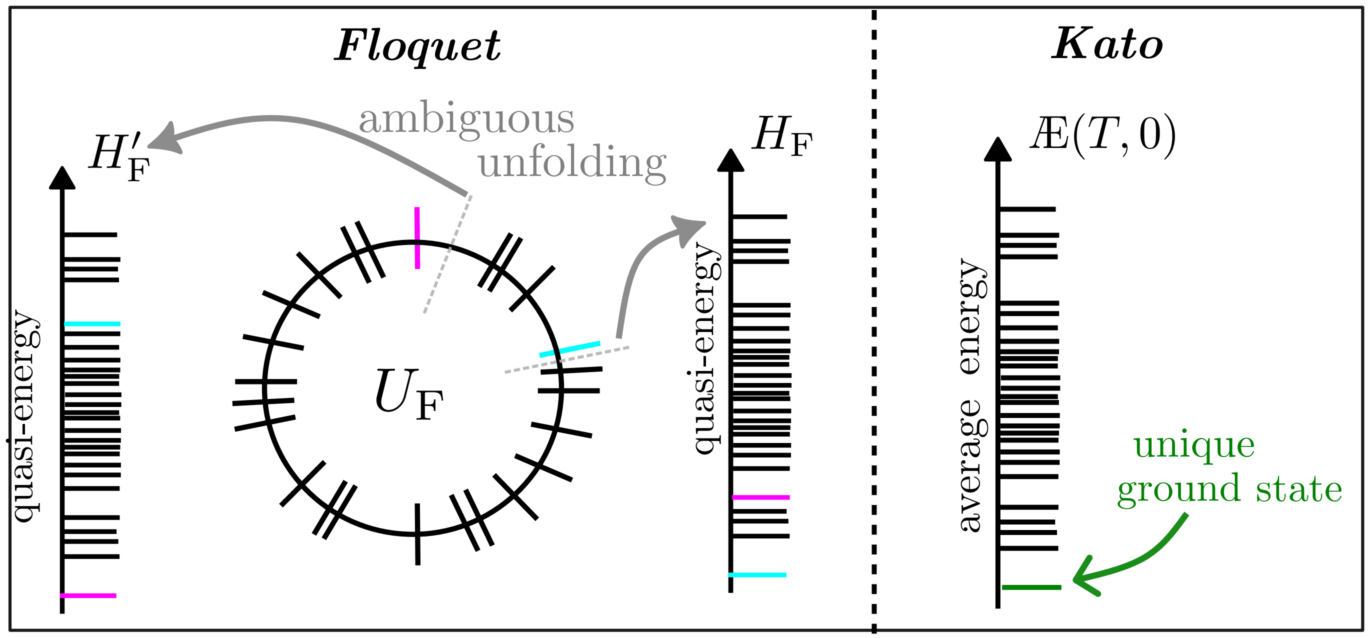

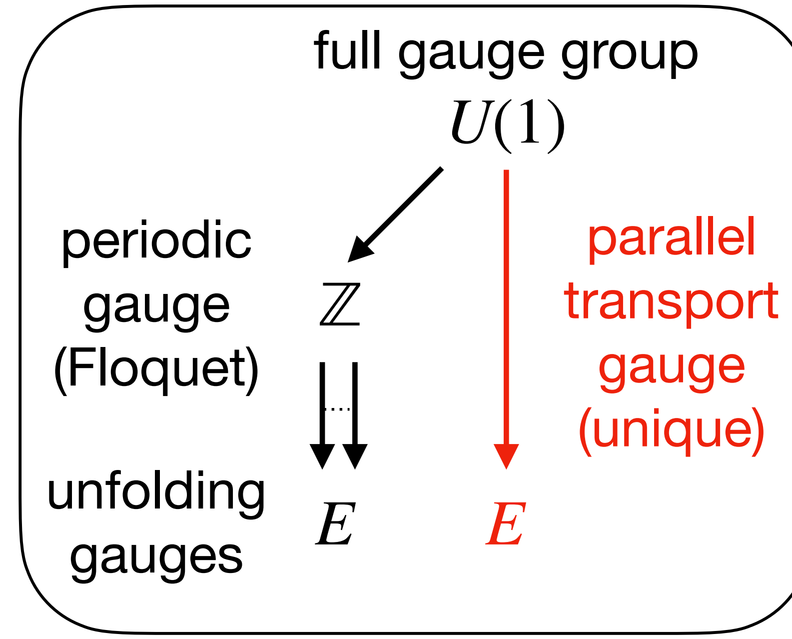

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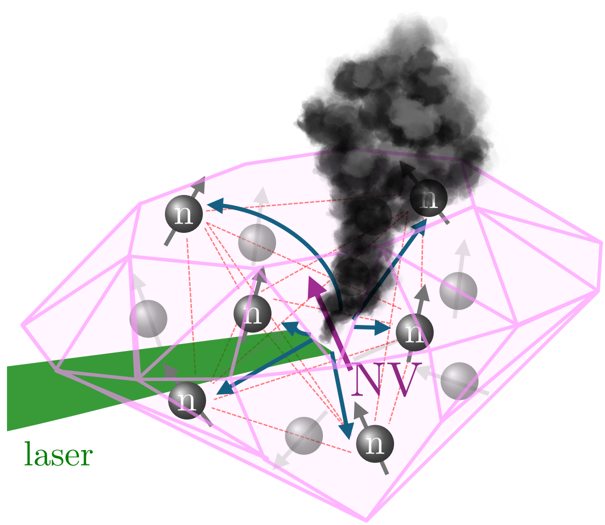
Outline

- **Applications**
  - heating, discrete time crystals
  - variational principle for Floquet Hamiltonian

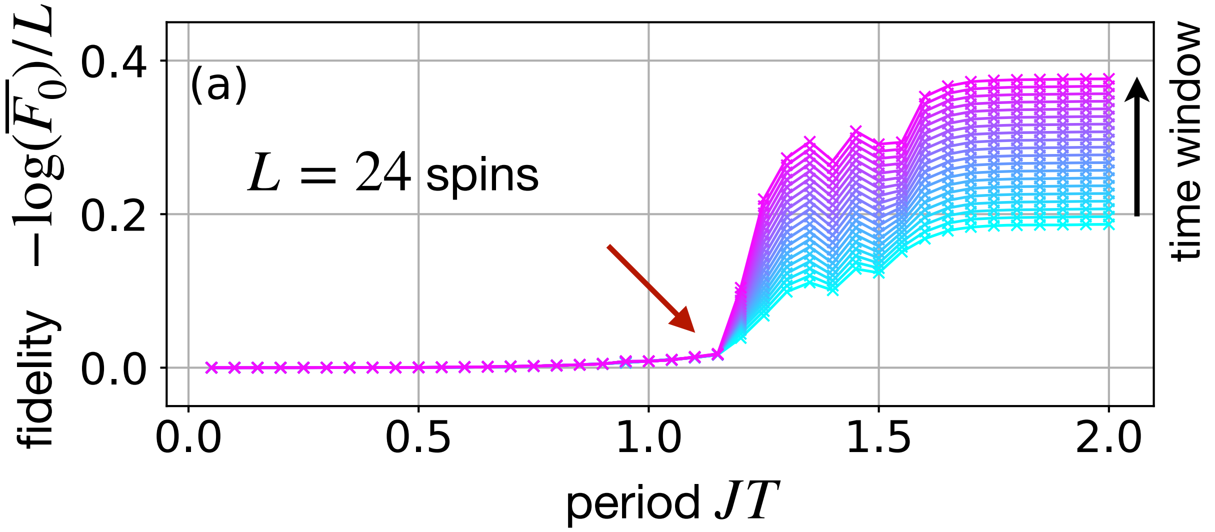


PM Schindler and MB, arXiv: 2410.07029

Heating in kicked Ising chain

- **evolution operator** $U_F = e^{-i\frac{T}{4}H^z} e^{-i\frac{T}{2}H^x} e^{-i\frac{T}{4}H^z}$
  - evolve GS $|\mathrm{GS}(t)\rangle$ of $H_F^{(0)} = H^x + H^z$
  - measure long-time fidelity with exact $|n_F\rangle$

$$F_0 = |\langle \mathrm{GS}(t)|n_F\rangle|^2$$



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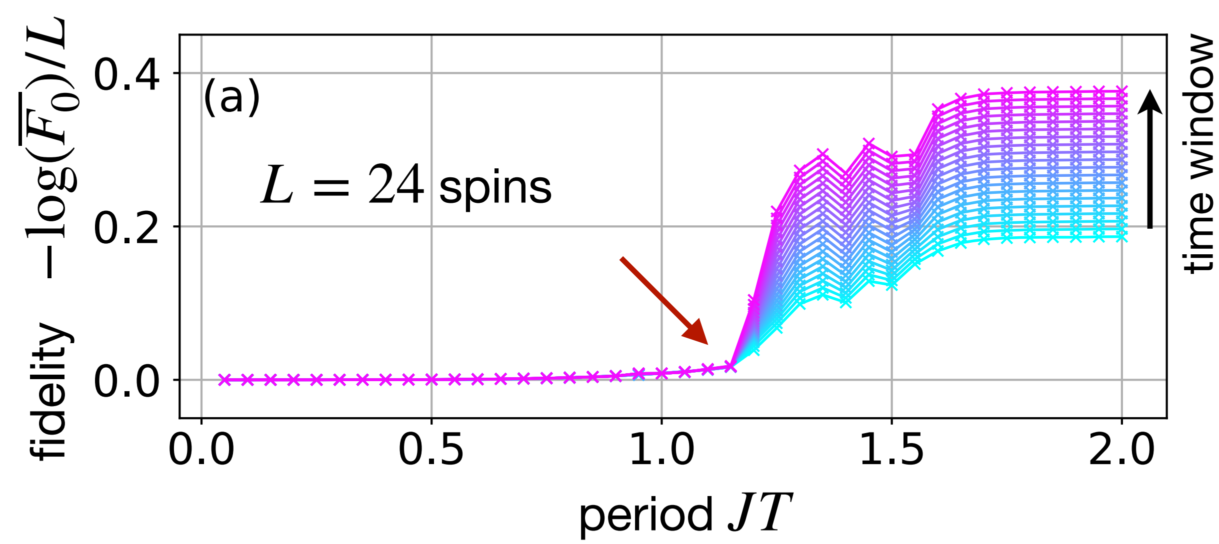
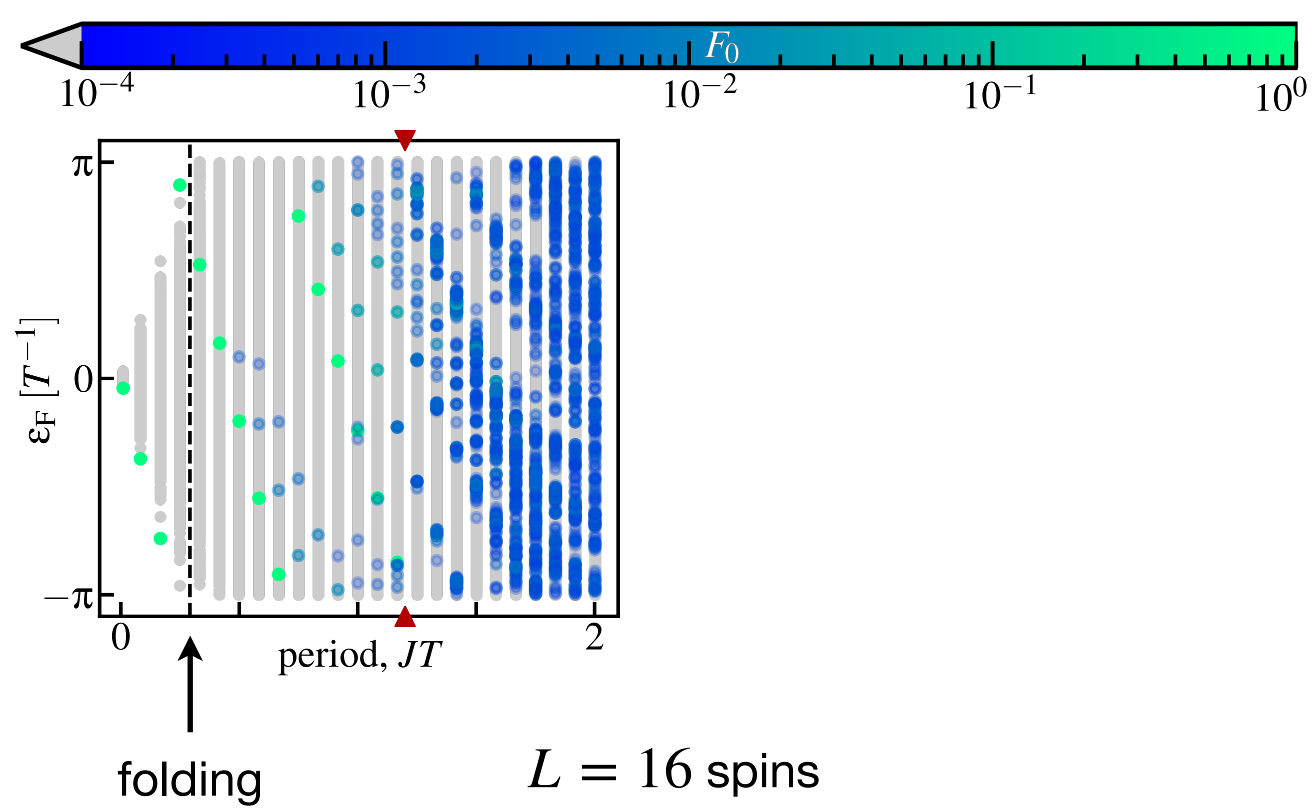


- **distribution over q'energy spectrum**
  - occupation gradually delocalizes



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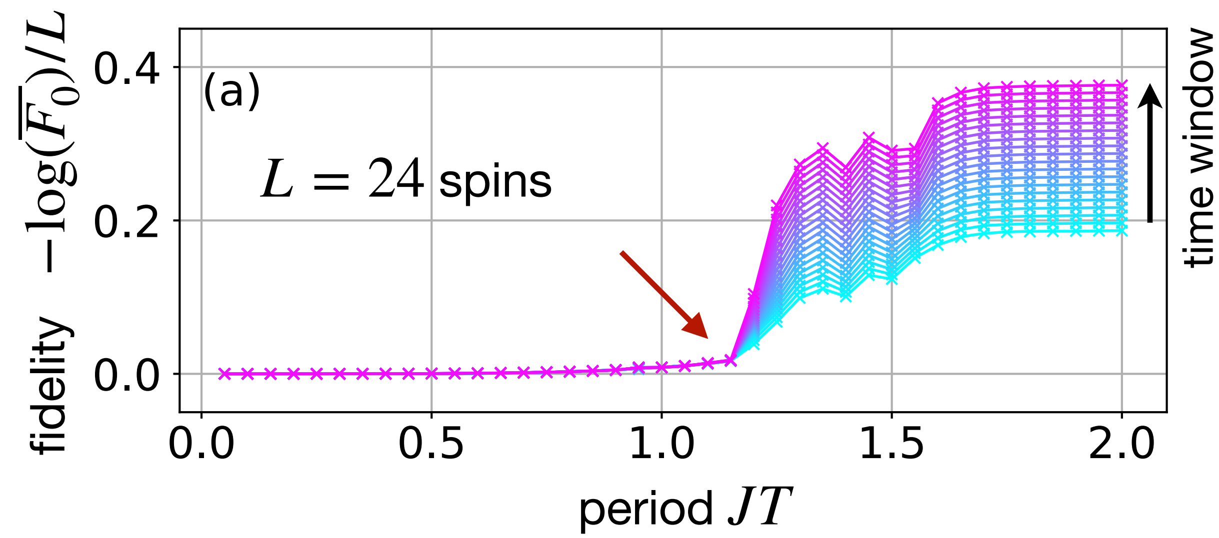
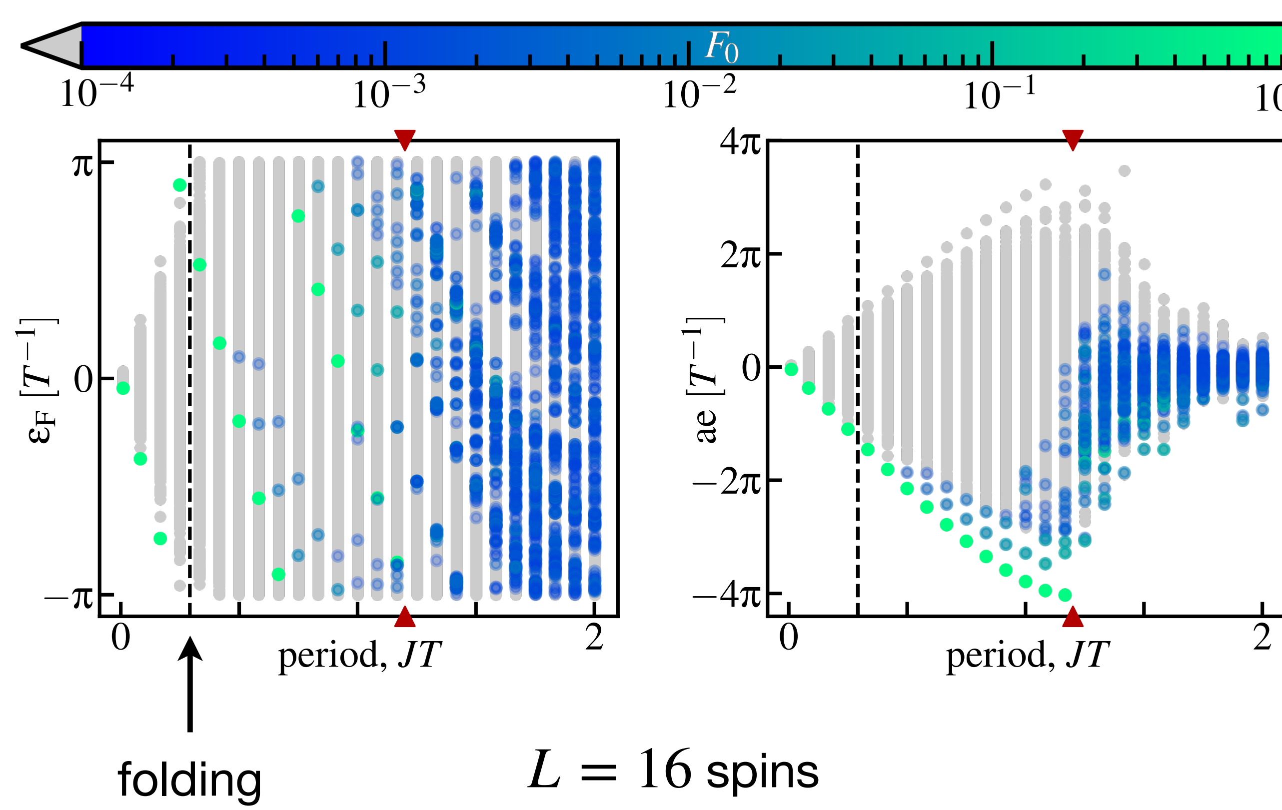


- **distribution over q'energy spectrum**
  - occupation gradually delocalizes
- **distribution over average energy**
  - occupation remains in Floquet GS
  - æ spectrum extensive up to T_*
  - æ spectrum implodes for $T > T_*$



Q: are certain Floquet states special?

Heating in kicked Ising chain

$$U_F = e^{-i\frac{T}{4}H^z} e^{-i\frac{T}{2}H^x} e^{-i\frac{T}{4}H^z}$$

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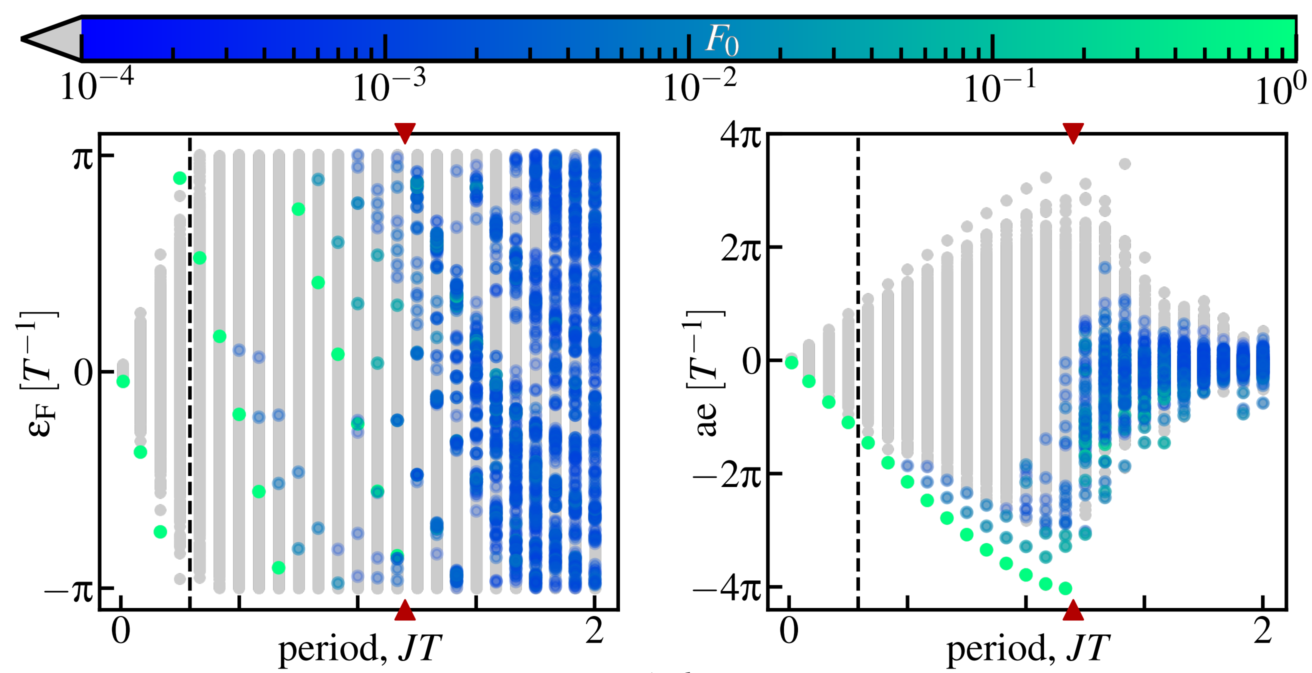

$L = 16$ spins

Q: are certain Floquet states special?

- **locality of average energy operator**

e'state projector

recall: H_F is non-local: $H_F = \sum_n \varepsilon_F^{(n)} |n_F\rangle\langle n_F| \longrightarrow \sum_n \varepsilon_F^{(n)} |n_F\rangle\langle n_F| + \omega |m_F\rangle\langle m_F| = H_F'$

Heating in kicked Ising chain

$$U_F = e^{-i\frac{T}{4}H^z} e^{-i\frac{T}{2}H^x} e^{-i\frac{T}{4}H^z}$$

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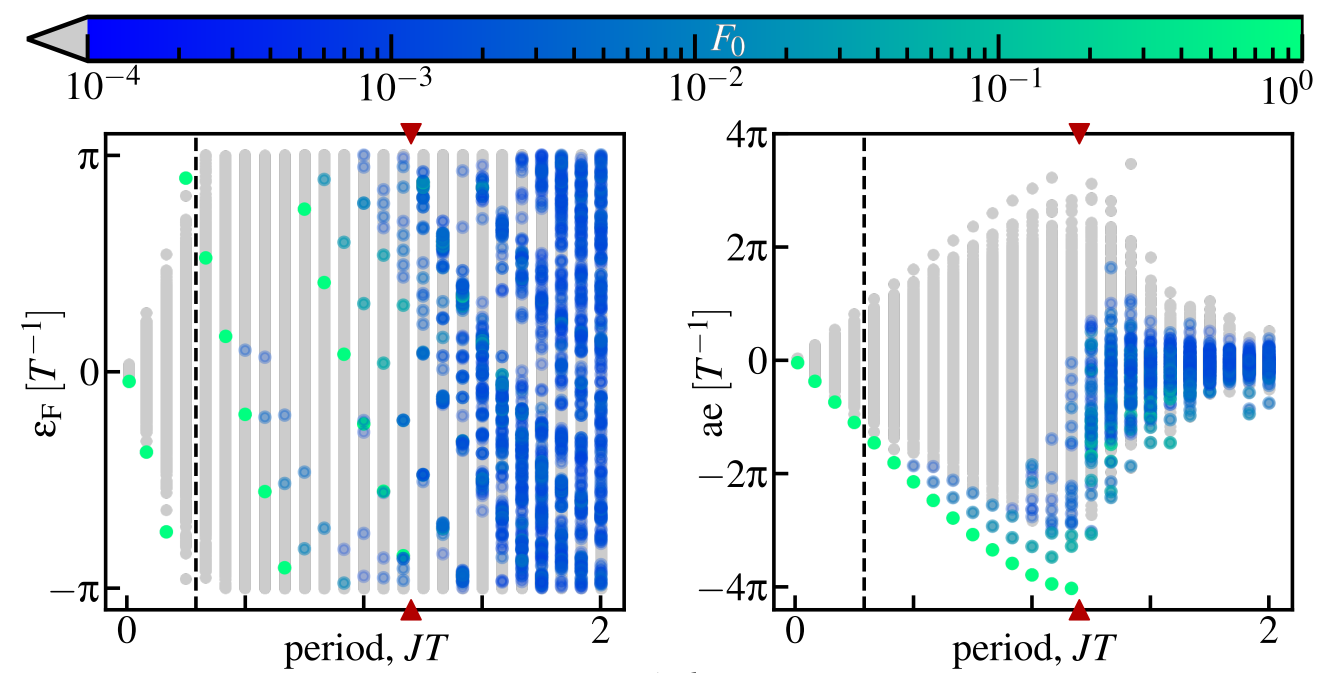

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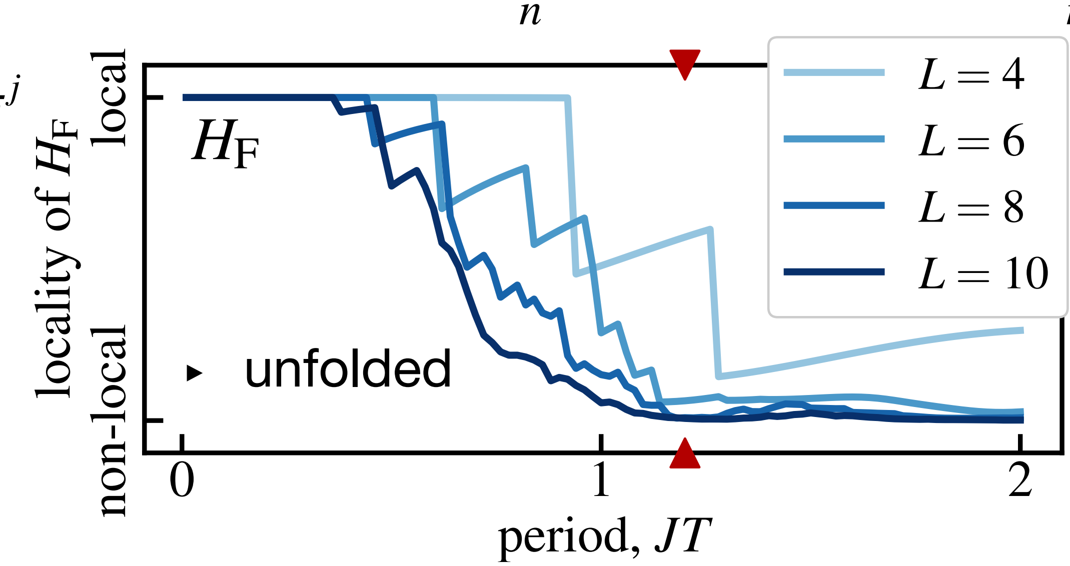
$$\mathcal{O}_{\text{approx}} = \sum_{i,j} o_i \sigma^i + o_{ij}\sigma^i\sigma^j$$

$$\frac{\|\mathcal{O}_{\text{approx}}\|}{\|\mathcal{O}_{\text{exact}}\|}$$



Heating in kicked Ising chain

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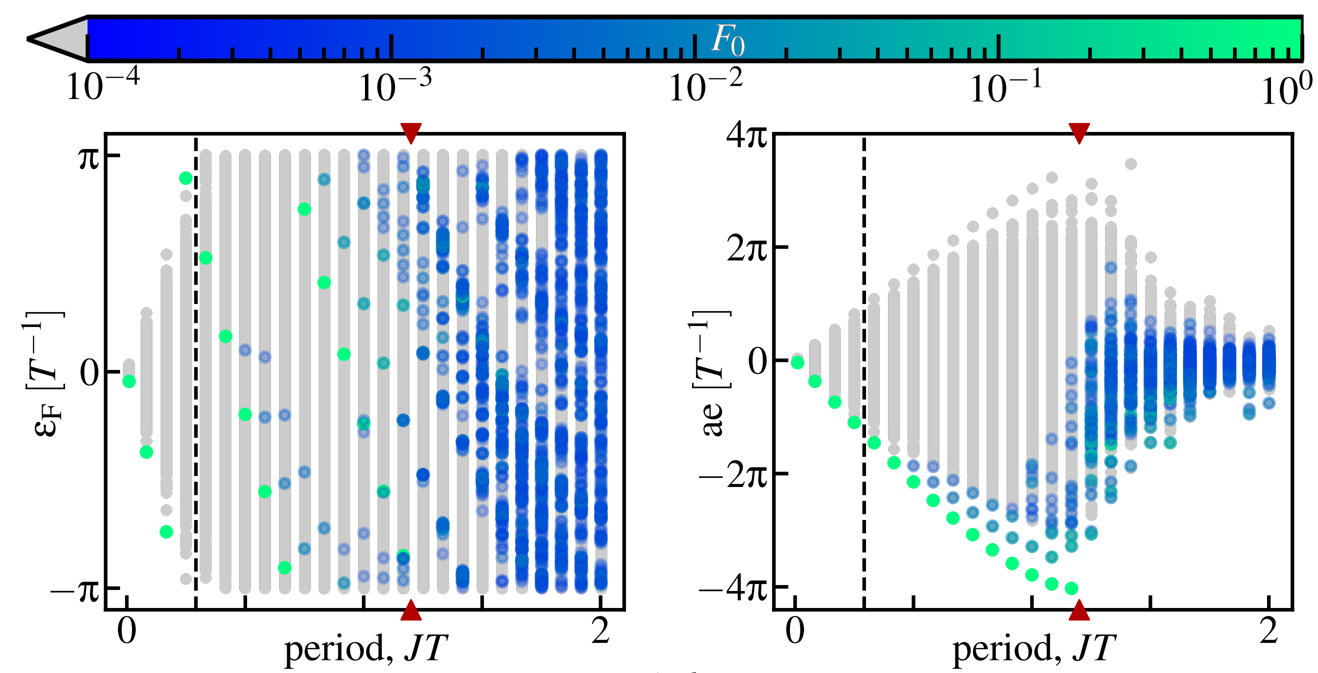

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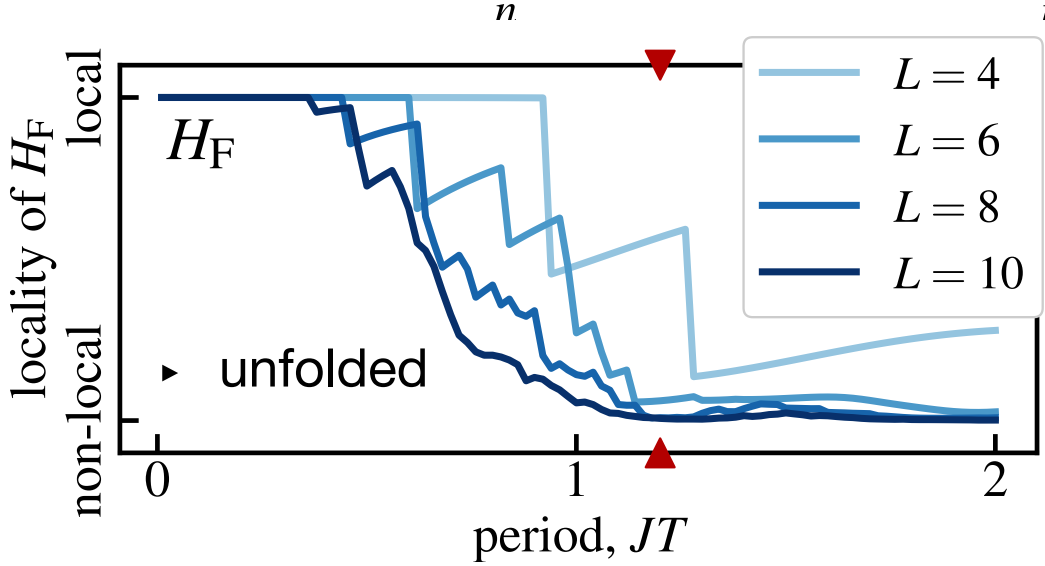
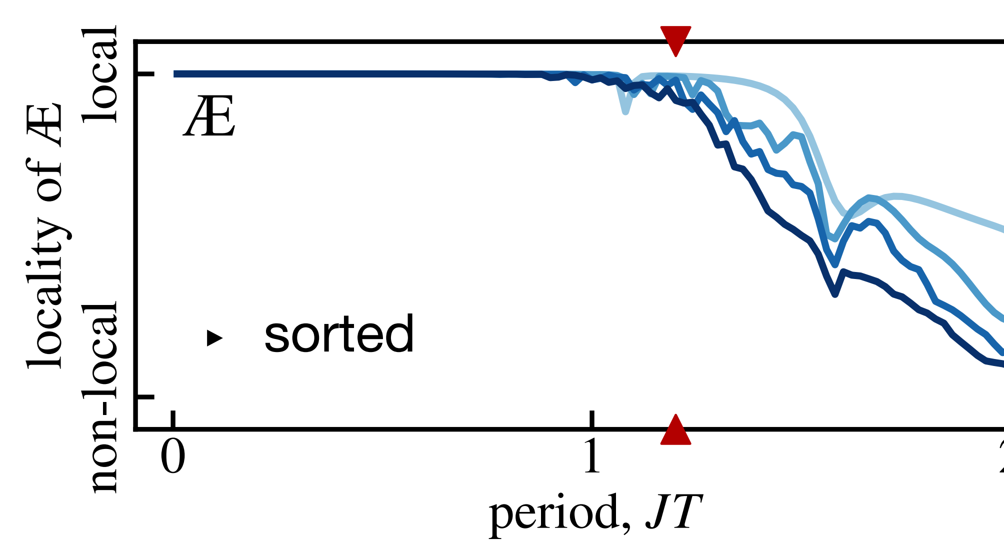
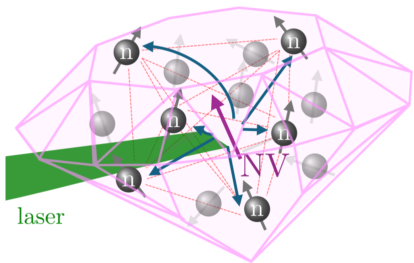


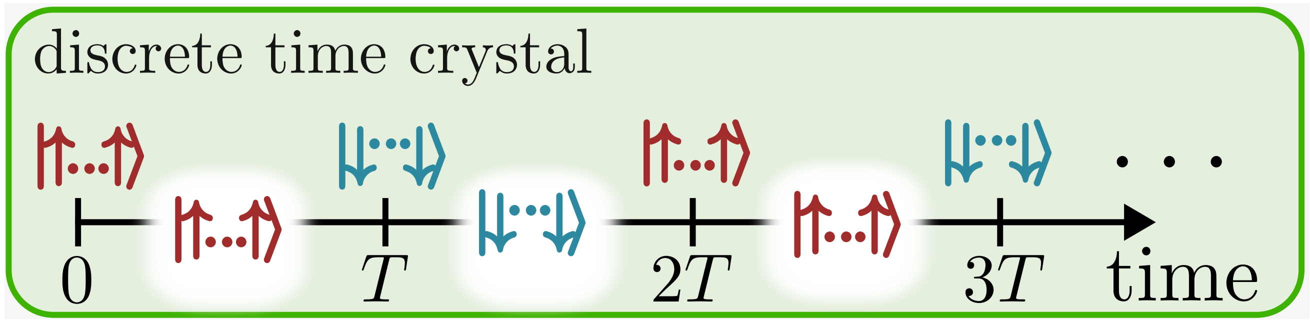


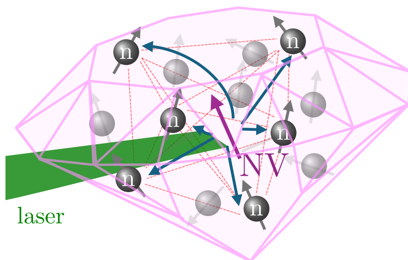
Q: is the average energy local?



Discrete time crystals

- **evolution operator** $U_F(\theta_x) = e^{-iTH^z} e^{-i\theta_x H^x}$





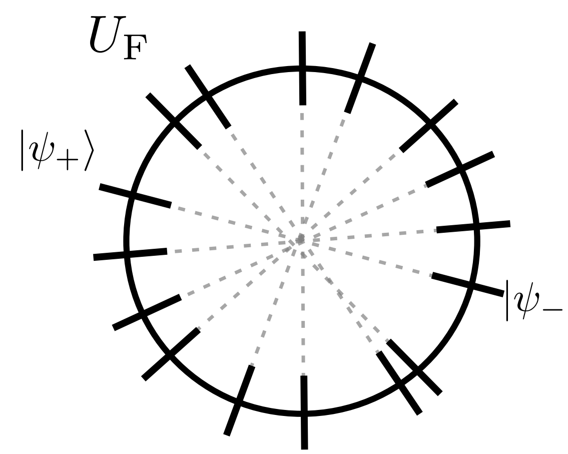
Discrete time crystals

- **evolution operator** $U_F(\theta_x) = e^{-iTH^z} e^{-i\theta_x H^x}$

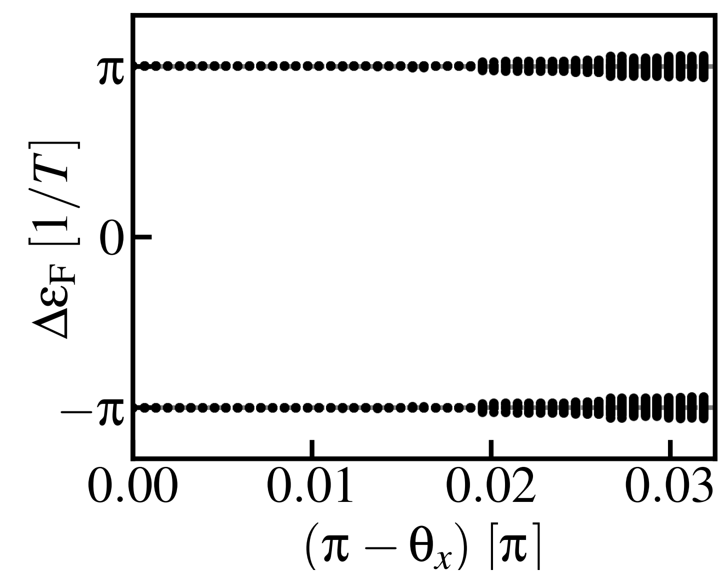
  - pairing of Floquet states

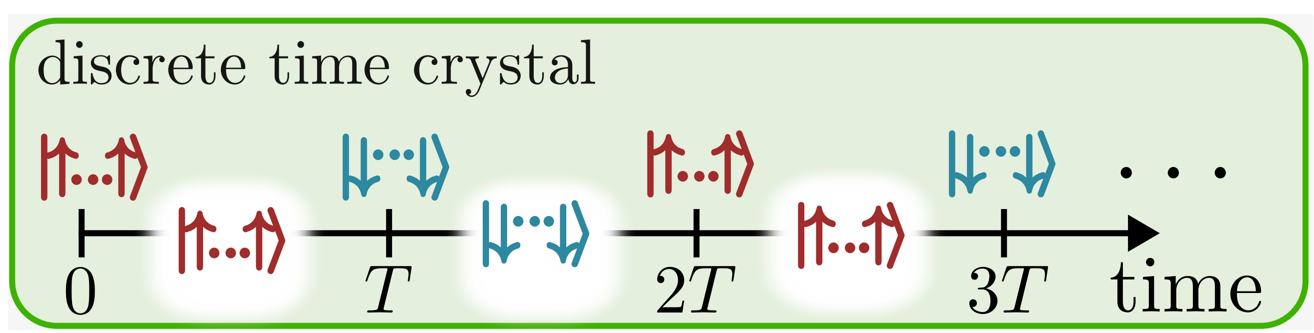
$$U_F(\pi) | n_F^{\pm} \rangle = \pm\, e^{-iT\varepsilon_n} | n_F^{\pm} \rangle$$

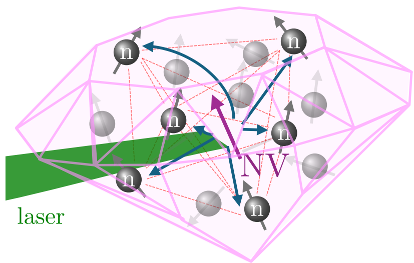
  - π-gap in q'energy spectrum



robust to perturbations in θ_x







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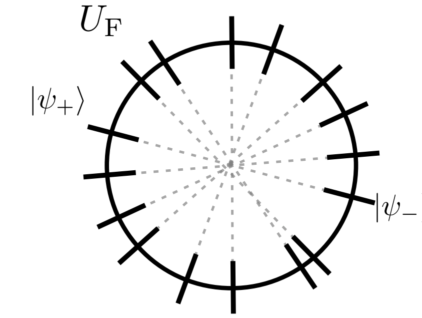

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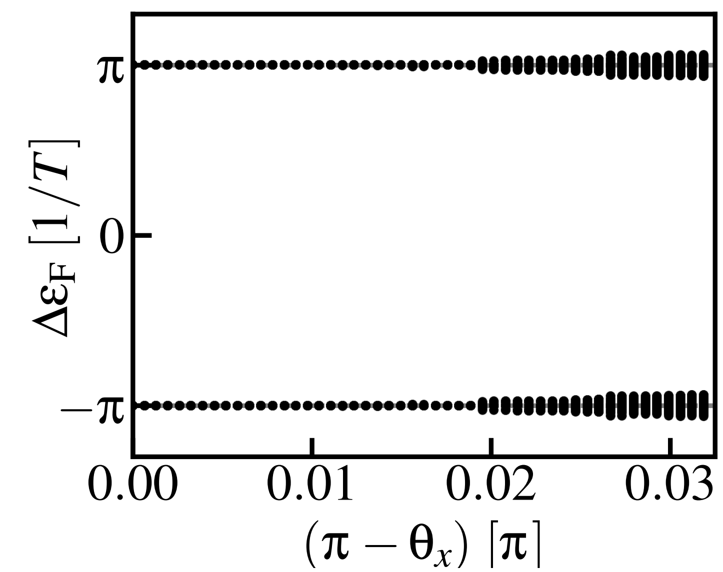
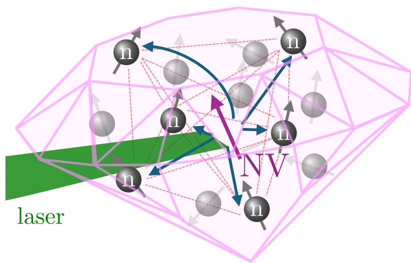


- **average energy**

$$æ_n = \frac{1}{t}\int_0^t \mathrm{d}s\, \langle n_F^\pm[s]\,|\,H(s)\,|\,n_F^\pm[s]\rangle = \varepsilon_n$$ perfectly degenerate!



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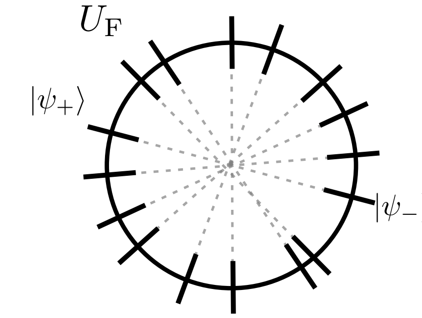

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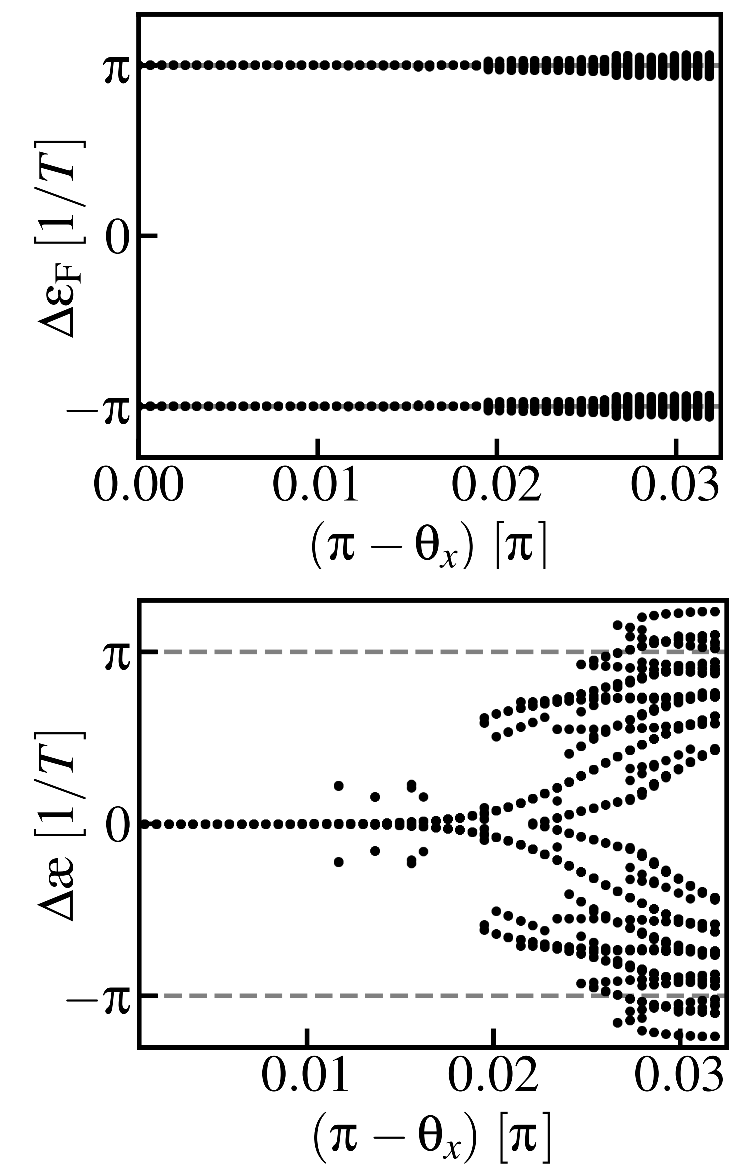

$L = 10$ spins

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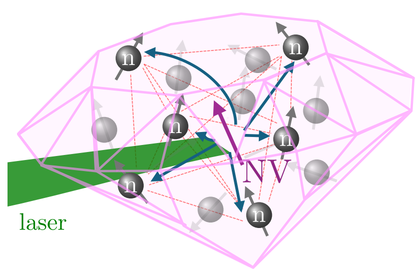
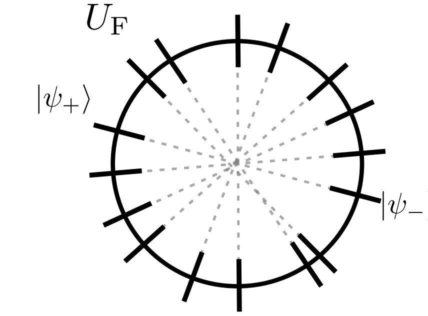


Discrete time crystals



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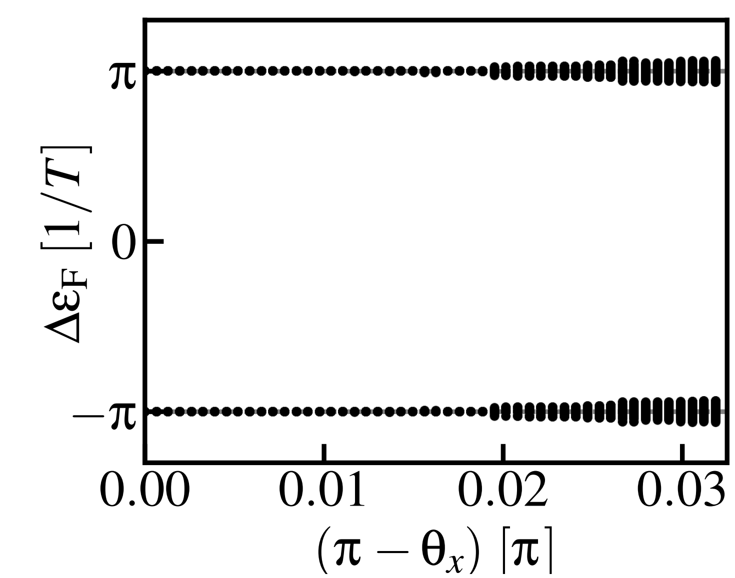

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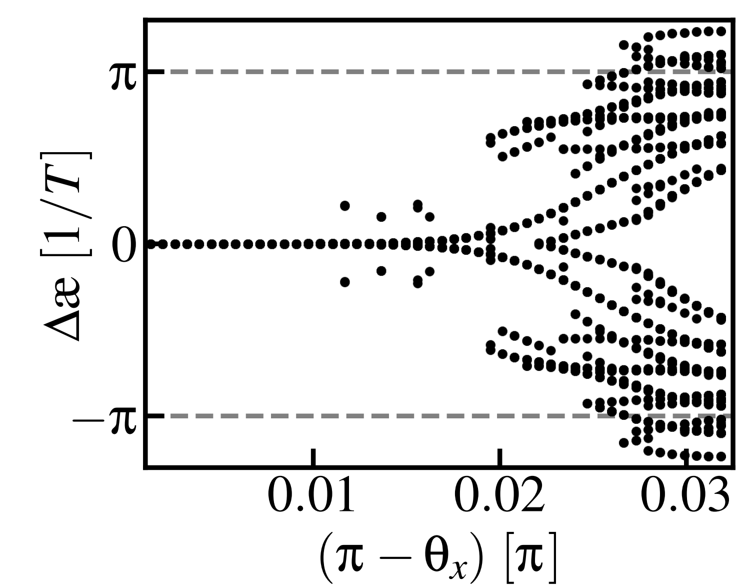
  - highly sensitive probe of DTC transition



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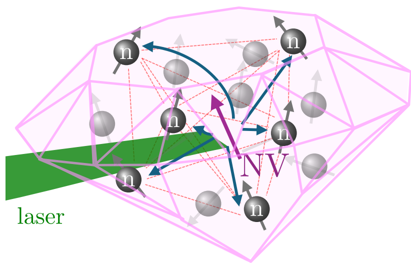

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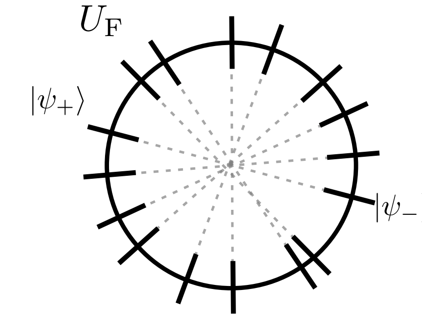

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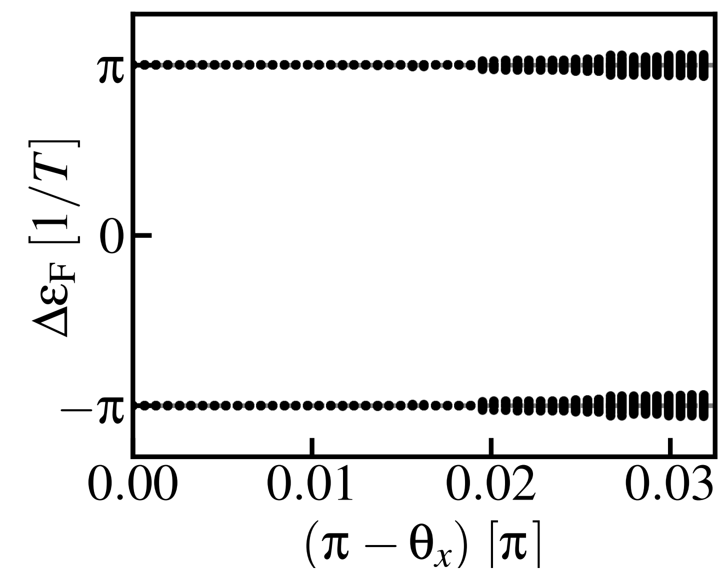

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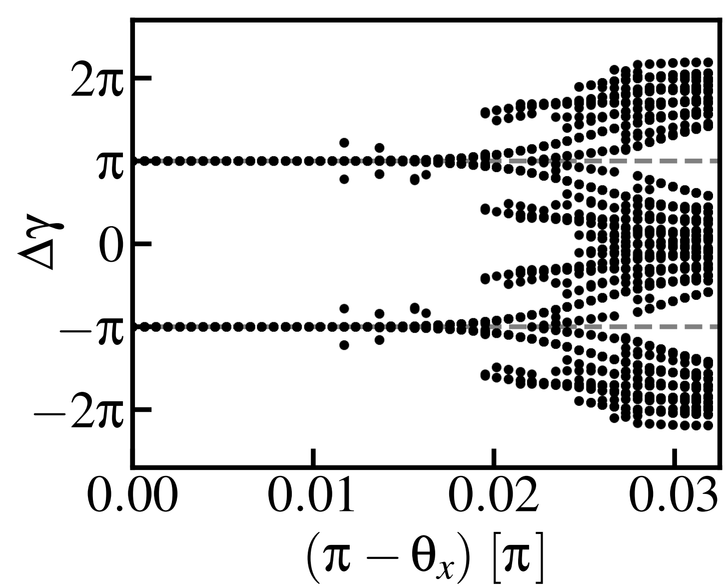
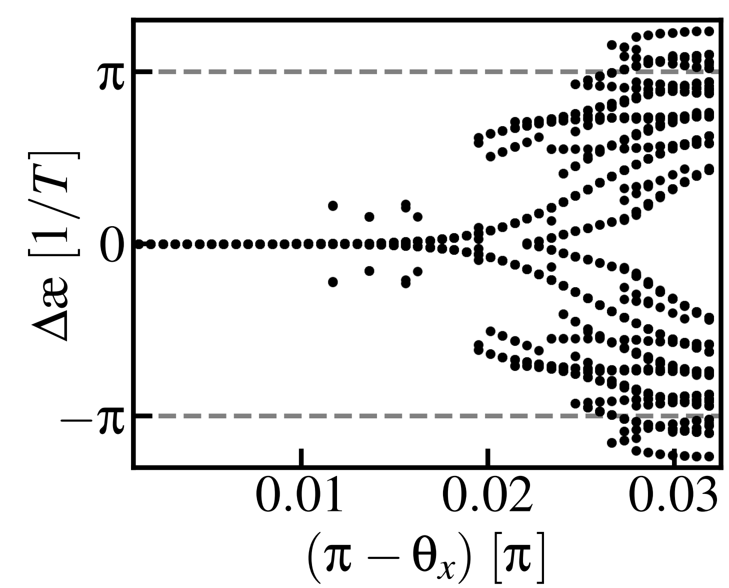
- **Berry phases**

$$\varepsilon_F^{(n)} = T^{-1}\gamma_n(T) + \text{æ}_n(T)$$

  - π-gap is purely geometric

→ similar for π-modes in AFTIs





$L = 10$ spins

❖ inherently nonequilibrium phenomena have geometric origin

Variational ansatz for H_F

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

- given drive $H(t)$, finding AGP $\mathscr{A}_F(t)$ determines Floquet Hamiltonian $H_F[t]$

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- turn to algorithm
  - make periodic ansatz for kick operator $K(t) = \sum_{n,\ell} k_{n\ell}\, e^{-i\ell\omega t} \mathscr{O}_n$ (unknown: $k_{n\ell}$; pre-selected: $\mathscr{O}_n$)

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  - update: $k_{n\ell} \to k_{n\ell} - \eta\,|\,g_{n\ell}\,|$, for some η

Variational ansatz for H_F

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

- **given drive $H(t)$, finding AGP $\mathscr{A}_F(t)$ determines Floquet Hamiltonian $H_F[t]$**
- **defining equation for gauge potential:** $G(\mathscr{A}_F) = i[H, \mathscr{A}_F] - \partial_t H + \partial_t \mathscr{A}_F = 0$
- **turn to algorithm**
  - make periodic ansatz for kick operator $K(t) = \sum_{n,\ell} k_{n\ell}\, e^{-i\ell\omega t} \mathscr{O}_n$ (unknown: $k_{n\ell}$; pre-selected: $\mathscr{O}_n$)
  - compute associated gauge potential: $\mathscr{X}_F(t) = \left(i\partial_t e^{iK(t)}\right) e^{-iK(t)}$
  - compute $G(\mathscr{X}_F) = \sum_{n,\ell} g_{n\ell}\, e^{-i\ell\omega t} \mathscr{O}_n$
  - update: $k_{n\ell} \to k_{n\ell} - \eta\,|\,g_{n\ell}\,|$, for some η
  - iterate until convergence

Variational ansatz for H_F

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

- **given drive $H(t)$, finding AGP $\mathscr{A}_F(t)$ determines Floquet Hamiltonian $H_F[t]$**

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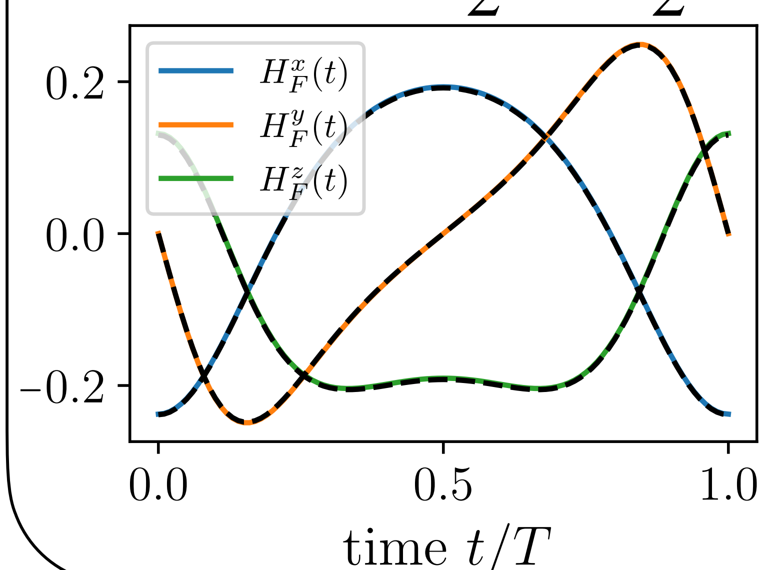
2LS: $H(t) = \frac{\Delta}{2}Z + \frac{g}{2}(1 + 2\cos\omega t)X$



resonant drive

$\omega/g = 2$

$\Delta/g = 2$

dashed: *numerics*

solid: *variational*

Variational ansatz for H_F

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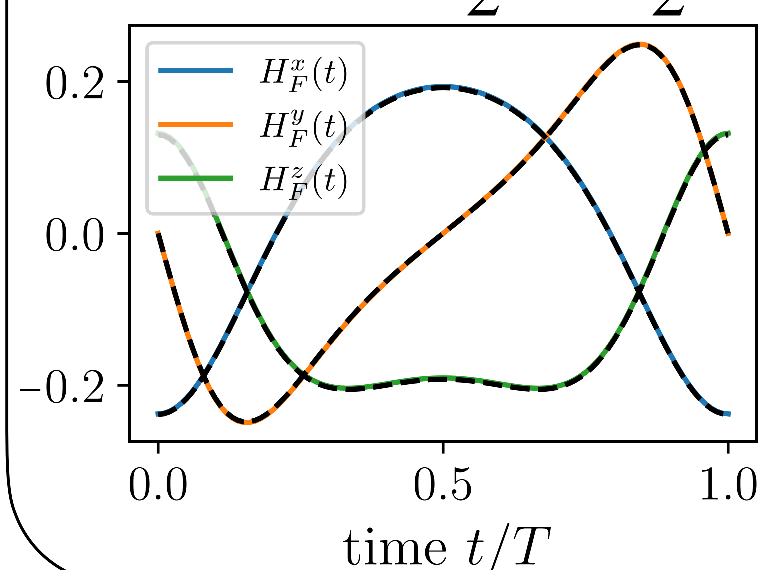
✓ no diagonalization

✓ no time-ordered exponentials

✓ no high-frequency regime

2LS: $H(t) = \frac{\Delta}{2} Z + \frac{g}{2}(1 + 2\cos\omega t) X$



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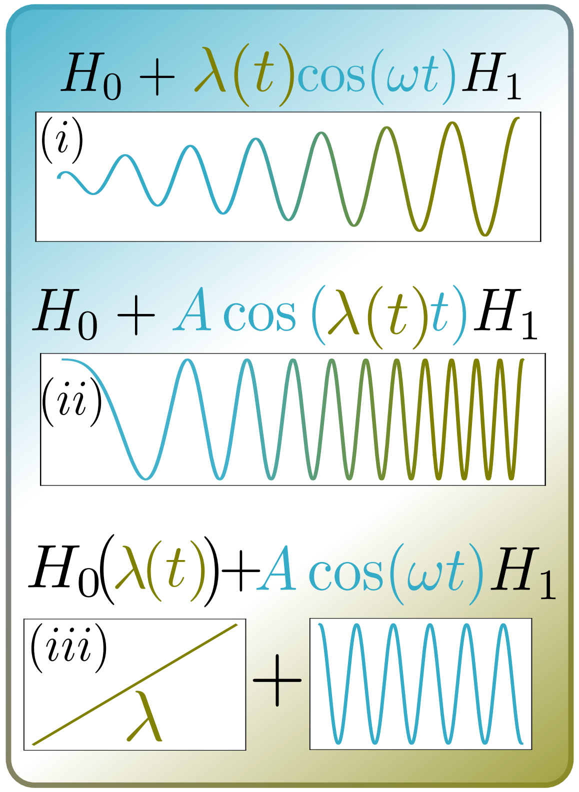
Controlling systems on top of Floquet drives

so far: ramp time/phase of the drive

- what about other control parameters?



amplitude ramps

frequency chirps

external ramps

- counterdiabatic driving of Floquet engineered states

PM Schindler and MB, PRL 133, 123402 (2024)

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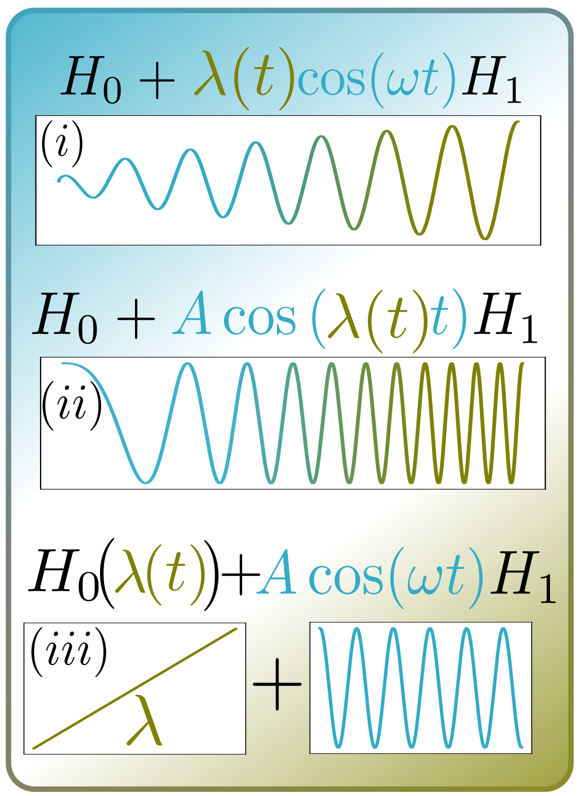

amplitude ramps

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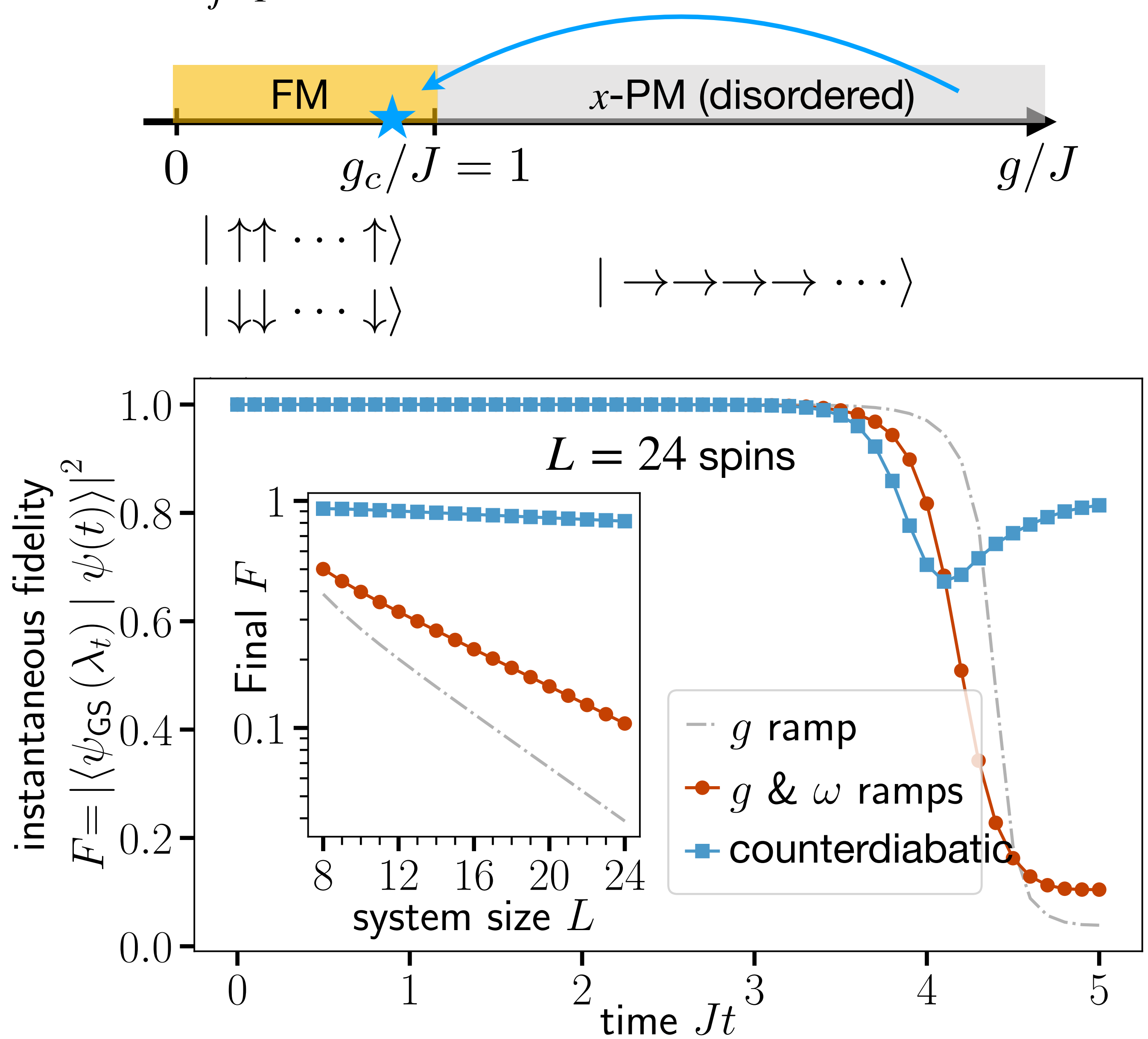
- counterdiabatic driving of Floquet engineered states

$$H(t) = \sum_{j=1}^{L} J\sigma^z_{j+1}\sigma^z_j + g(t)\left[\cos(\varphi(t))\sigma^x_j + \sin(\varphi(t))\sigma^y_j\right]$$



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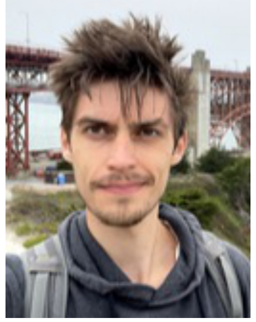

Paul M Schindler

Summary & Outlook

PM Schindler and MB, arXiv: 2410.07029

www.pks.mpg.de/nqd

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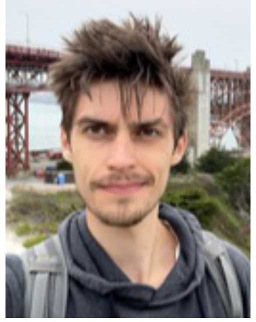
  - lab frame Hamiltonian $H(t)$ generates CD driving for Floquet Hamiltonian $H_F[t]$
  - parallel-transport formulation leads to unique Floquet ground state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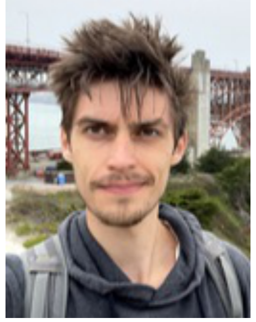

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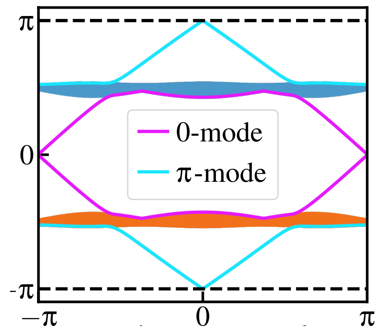

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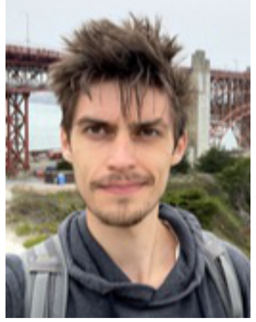

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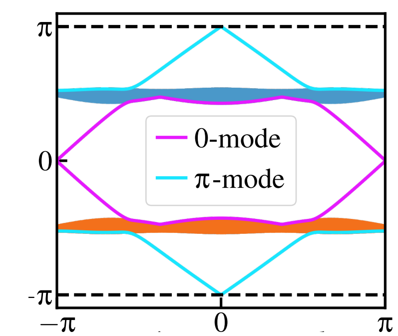

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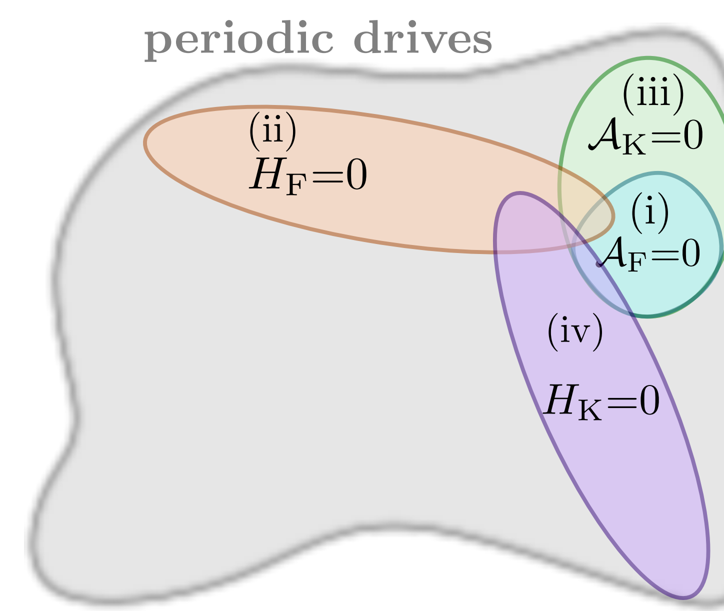
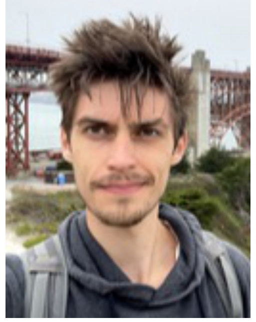
- *Kato decomposition:* $H(t) = H_K(t) + \mathscr{A}_K(t)$





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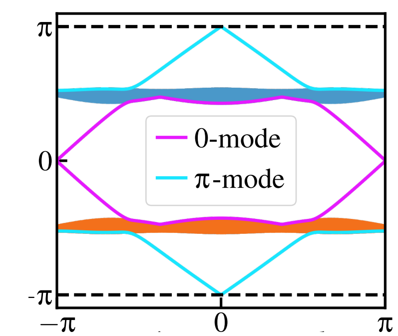

◉ **'elementary' families of periodic drives:**

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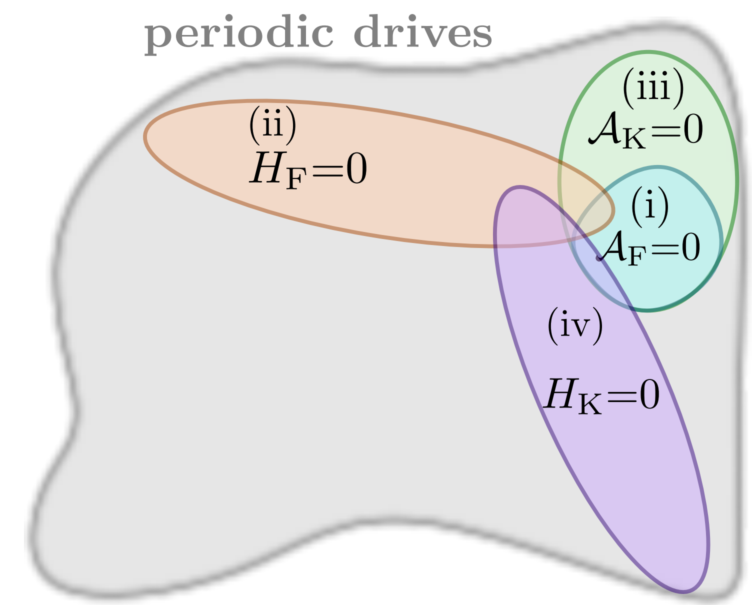
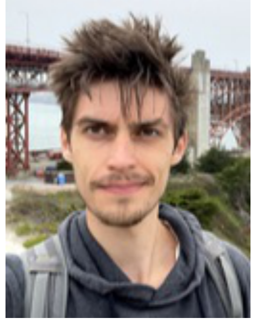
  - (ii) pure-micromotion drives: $H_F \equiv 0 \implies U(t) = P(t)$ **no heating**
- *Kato decomposition:* $H(t) = H_K(t) + \mathcal{A}_K(t)$





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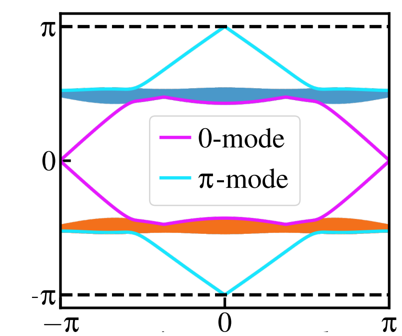

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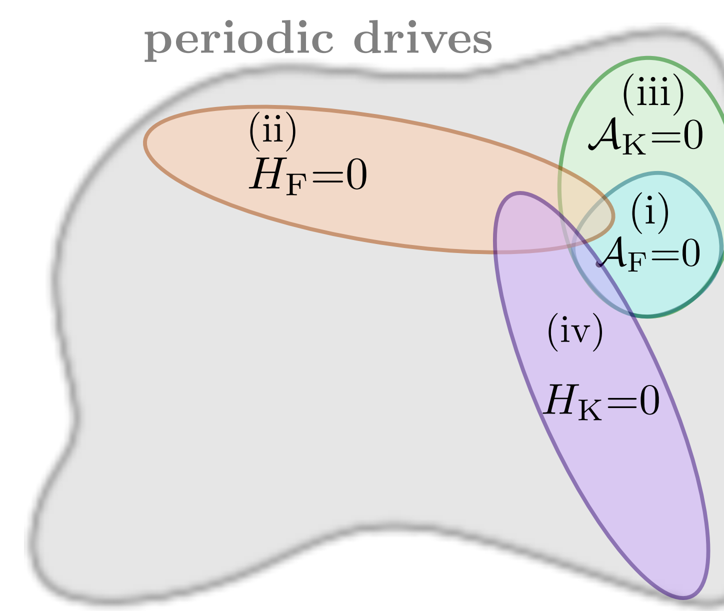
  - *Kato decomposition:* $H(t) = H_K(t) + \mathscr{A}_K(t)$
    - (iii) flat drives: Æ
    - (iv) pure-geometric drives: $H_K \equiv 0 \implies U(t) = W(t)$ q'energy = geometric phase (no Floquet ground state!)



Floquet resonances

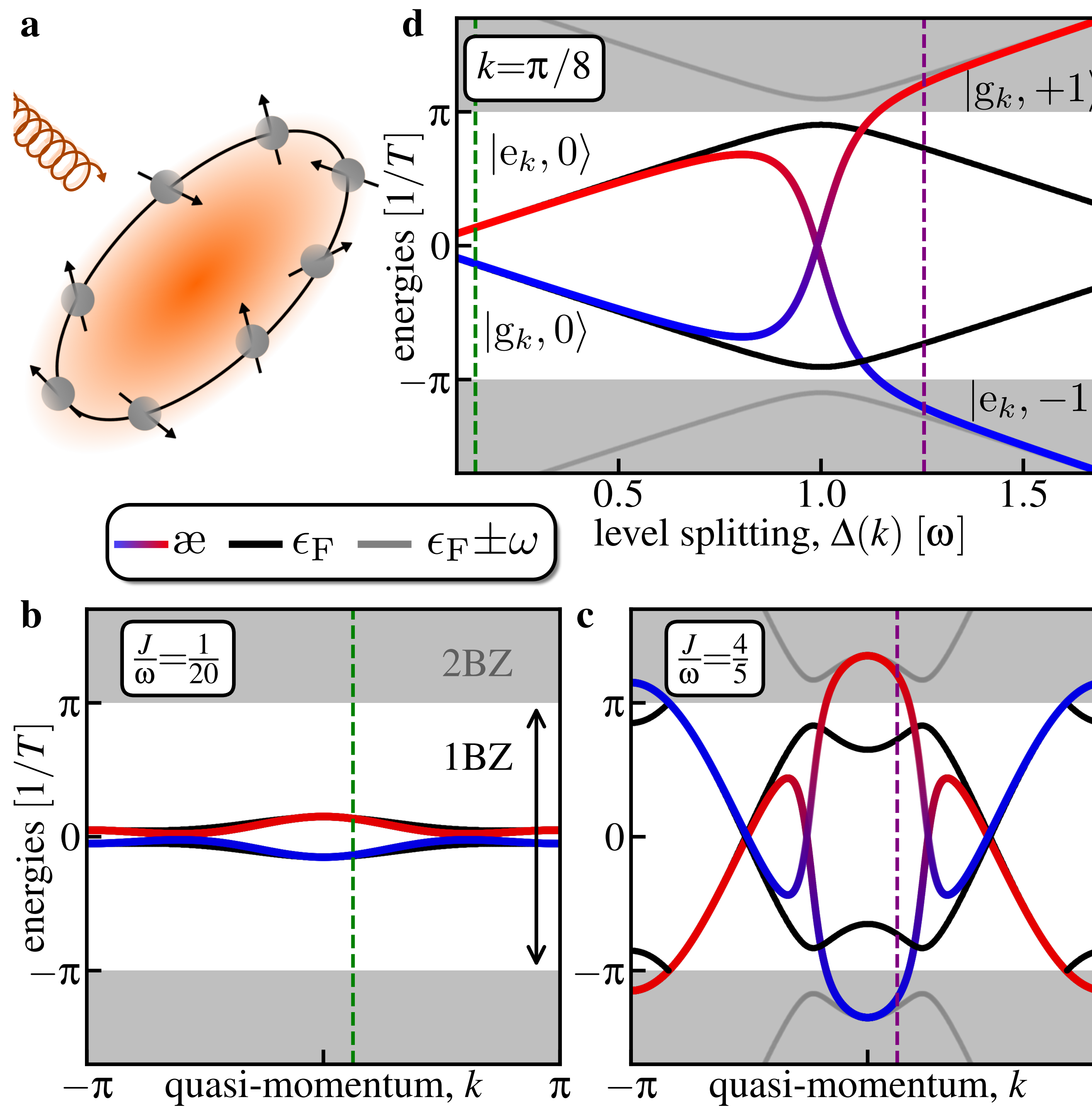
$$H(t) = \frac{1}{2}\sum_{n=1}^{L}\left[\left(J\sigma^+_{n+1}\sigma^-_n + Aie^{-i\omega t}\sigma^+_{n+1}\sigma^+_n + \mathrm{h.c.}\right) + \frac{g}{2}\sigma^z_n\right]$$

$$H(t) = \sum_k \boldsymbol{\psi}_k^\dagger h(k,t)\boldsymbol{\psi}_k$$

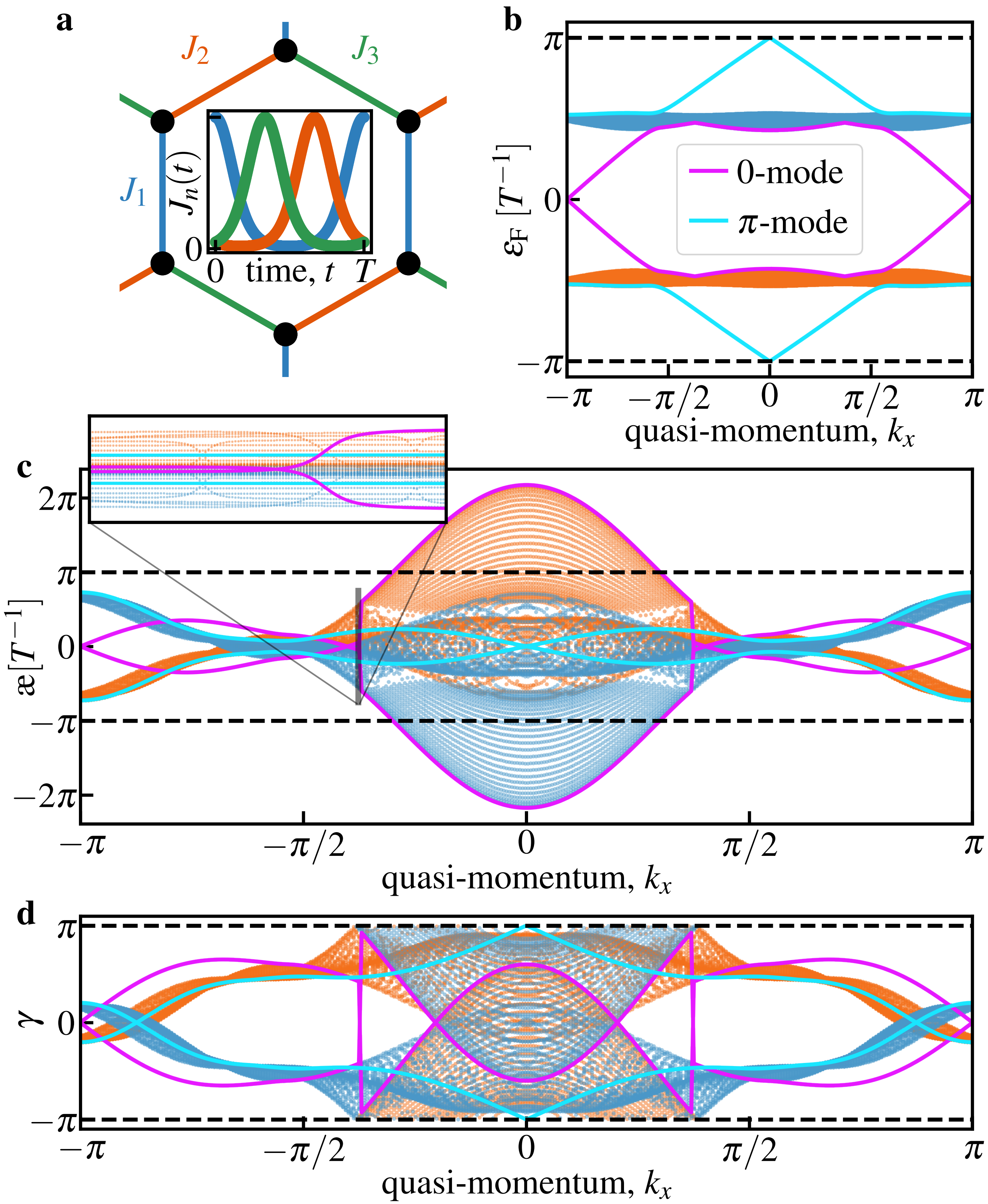
$$h(k,t) = \Delta_k\tau^z + A_k\left[\cos(\omega t)\tau^x + \sin(\omega t)\tau^y\right]$$

$$\Delta_k = g + J\cos(k)$$

$$A_k = A\sin(k)$$



Anomalous Floquet topological insulators



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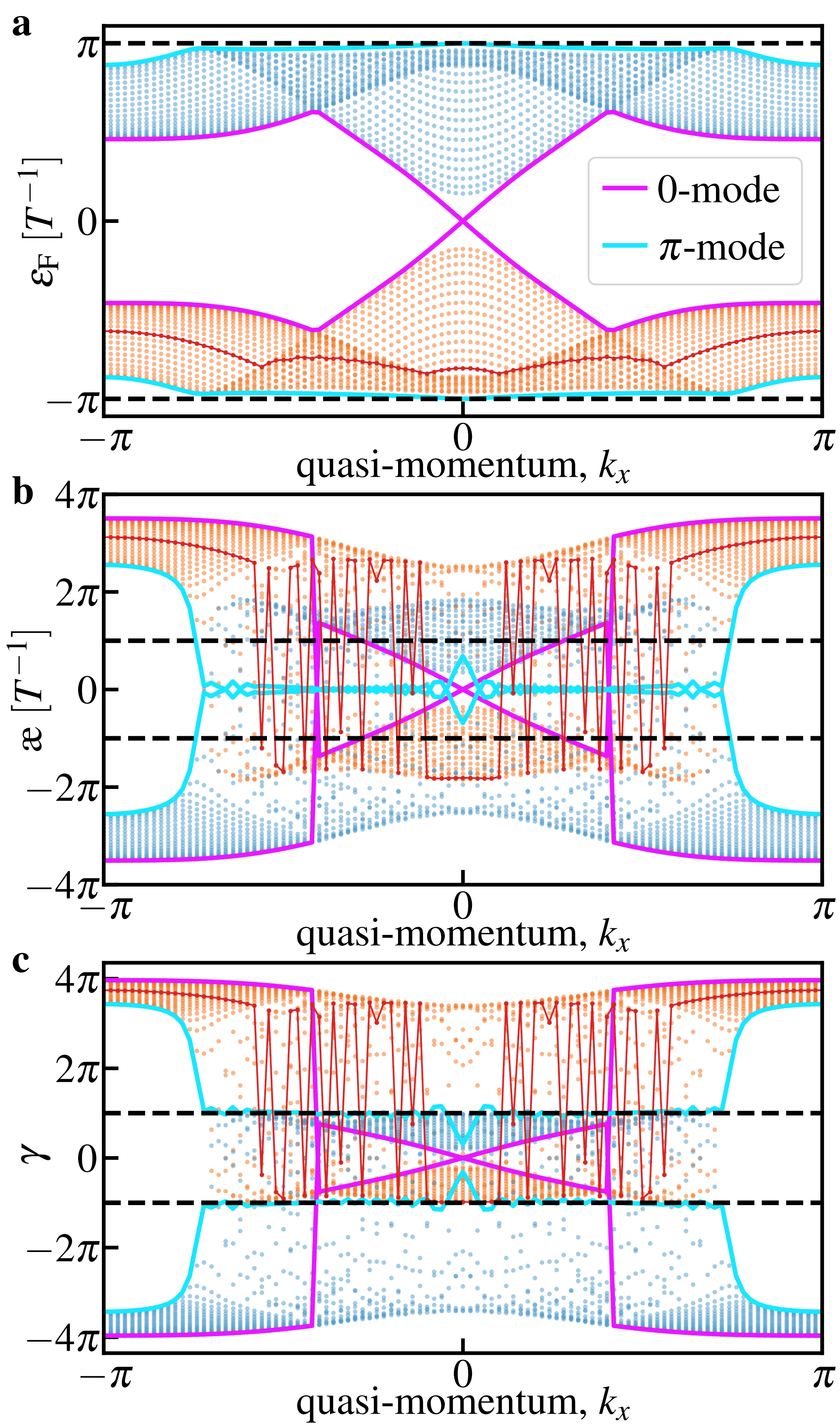
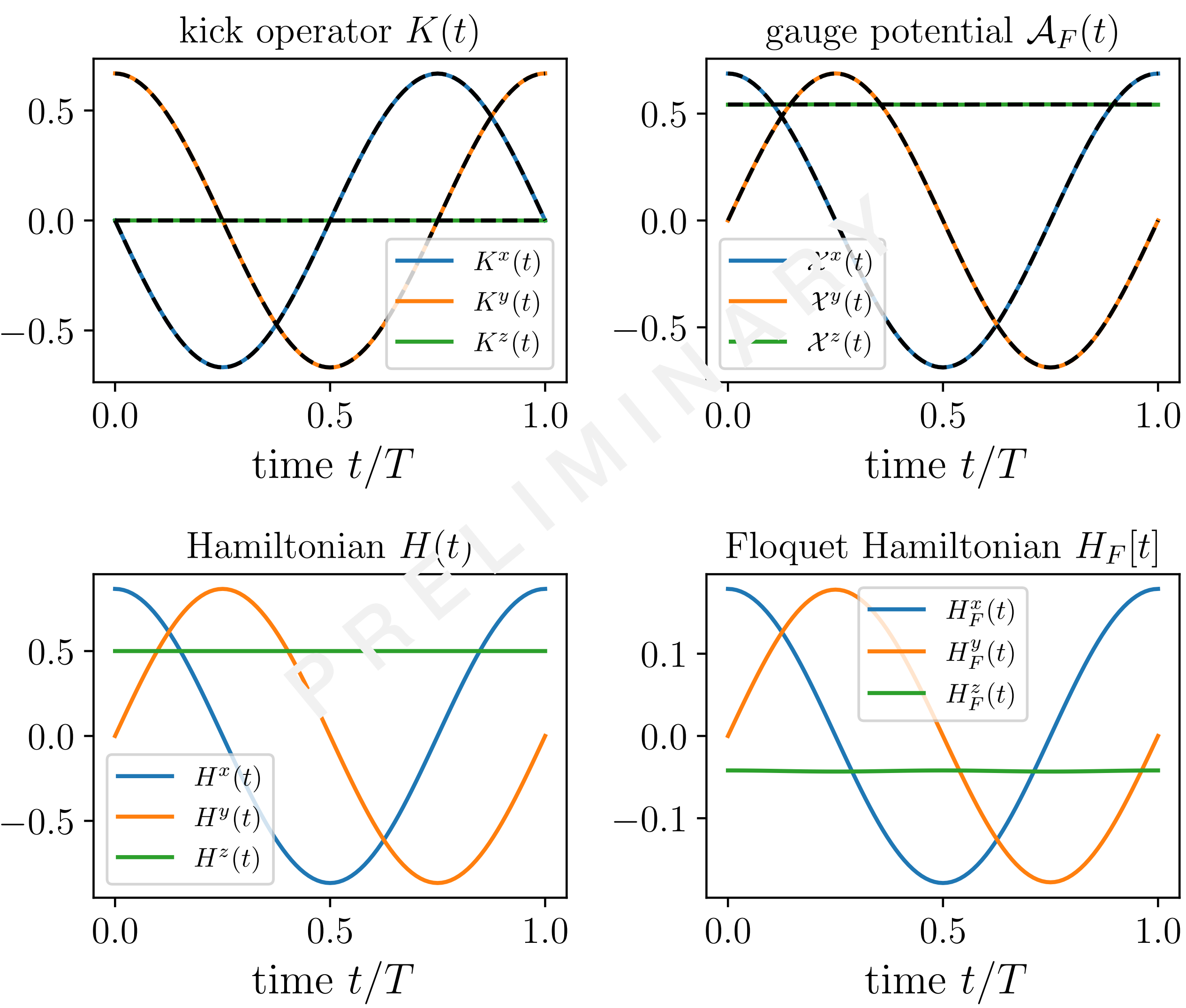


Variational approximation of H_F

- 2LS: circular drive

$$H(t) = \frac{1}{2}\Delta Z + \frac{g}{2}(\cos\omega t X + \sin\omega t Y)$$

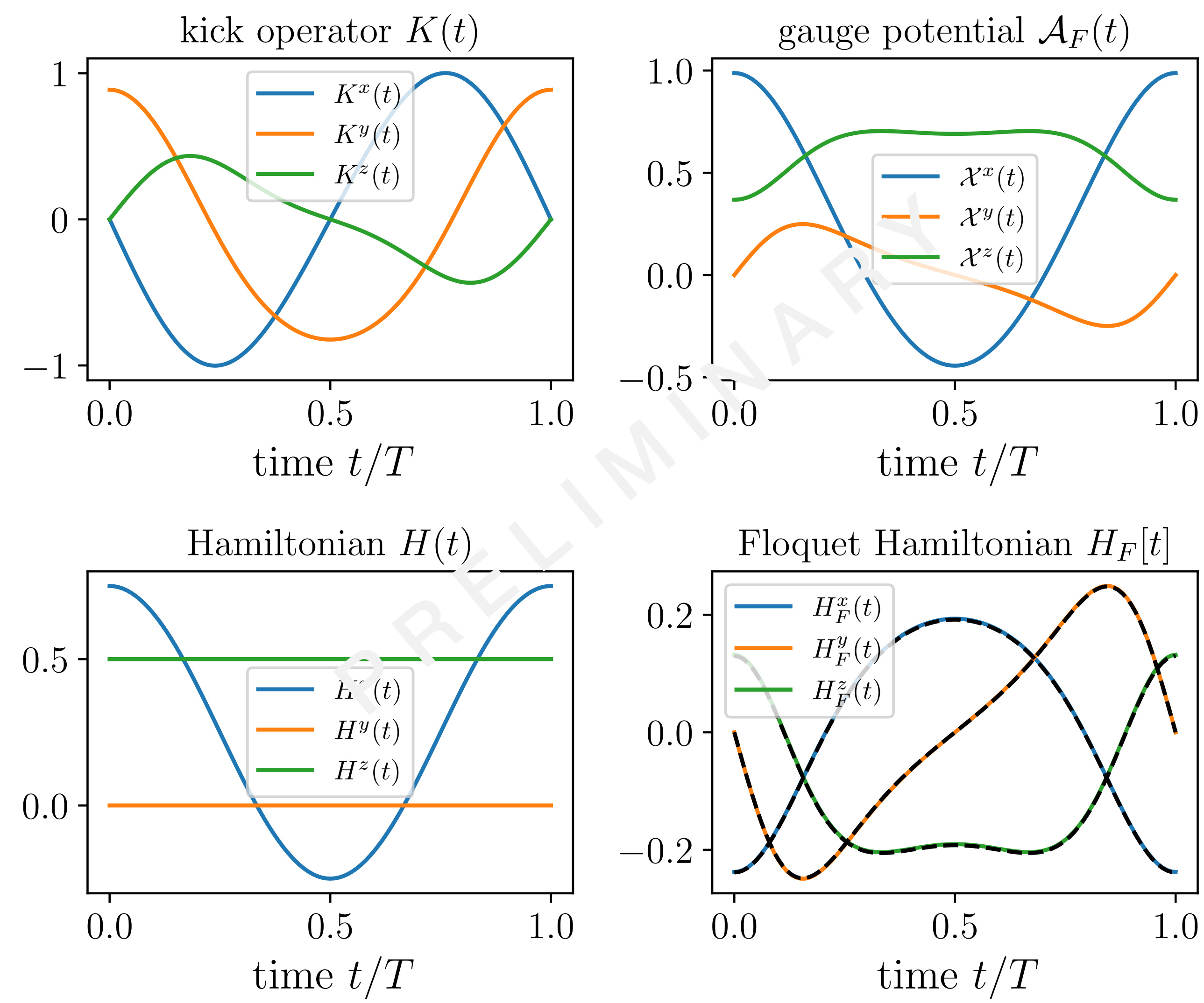


✓ matches exact solution (dashed line)

$$\omega/g = \sqrt{2}, \quad \Delta/g = 1$$

- 2LS: *resonant* linear drive

$$H(t) = \frac{1}{2}\Delta Z + \frac{g}{2}(1 + 2\cos\omega t)X$$



✓ agrees with numerics (dashed line)

$$\omega/g = 2, \quad \Delta/g = 2$$

21 harmonics

Variational approximation of H_F

- **nonintegrable Ising chain:** $H(t) = \sum_j JZ_{j+1}Z_j + h_z Z_j + h_x \sin \omega t X_j$

- **numerically compute exact H_F :** ground truth

- **compute approximation to H_F**
  - numerically, variational $\mathscr{H}_F$

$$K \in \left\{ \sum_j X_j, \sum_j Y_j, \sum_j Z_j, \sum_j X_j X_{j+1}, \sum_j Y_j Y_{j+1}, \sum_j Z_j Z_{j+1}, \sum_j X_j Y_{j+1} + Y_j X_{j+1}, \sum_j Y_j Z_{j+1} + Z_j Y_{j+1}, \sum_j Z_j X_{j+1} + X_j Z_{j+1} \right\}$$

+ keep up to 21 Fourier harmonics

  - analytically, Floquet-Magnus $H_{\mathrm{FM,n}}$ (to a fixed order $n = 0,1,2$)

$$H_{\mathrm{FM}}^{(0)} = \frac{1}{T}\int_0^T \mathrm{d}t H(t) \qquad H_{\mathrm{FM}}^{(1)} = \frac{1}{2!Ti}\int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, [H(t_1), H(t_2)] \qquad \cdots$$

- **compare time evolution operators:** $||e^{-iTH_F} - e^{-iT\mathscr{H}_F}||, \ ||e^{-iTH_F} - e^{-iTH_{\mathrm{FM}}^{(n)}}||$

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$$\|A - B\|^2 = 1 - \frac{1}{\dim(H)} \mathrm{Re}\ \mathrm{tr}(A^\dagger B)$$

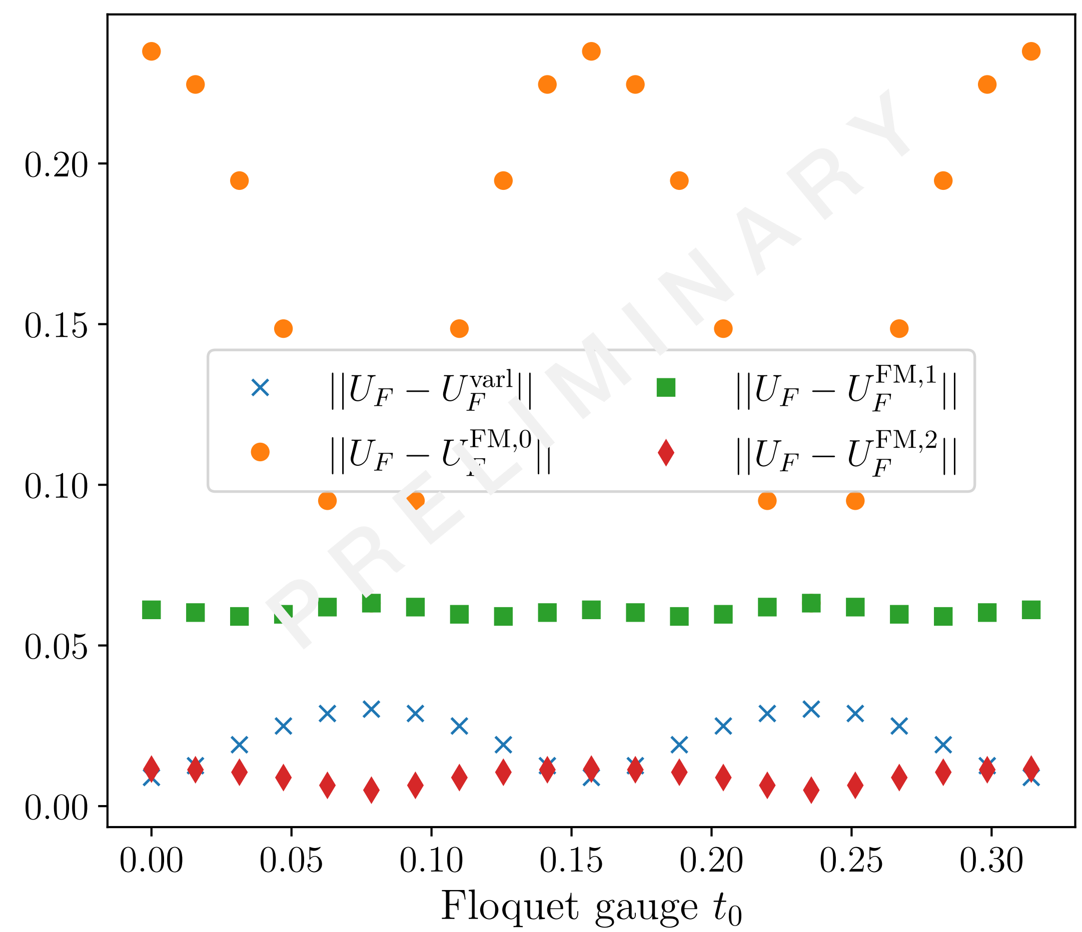
$$\|A - B\|^2 \in [0,2]$$

$$U_F = e^{-iTH_F}$$

$$\|e^{-iTH_F} - e^{-iT\mathscr{H}_F}\|$$

$$\|e^{-iTH_F} - e^{-iTH^{(n)}_{\mathrm{FM}}}\|$$

high frequency regime



$\omega/J = 20, \quad h_x/J = \sqrt{3}, \quad h_z/J = 0.9$

- 2nd order FM: $[ZZ, [X, ZZ]] \sim ZXZ$, etc.

12 spins, 21 harmonics

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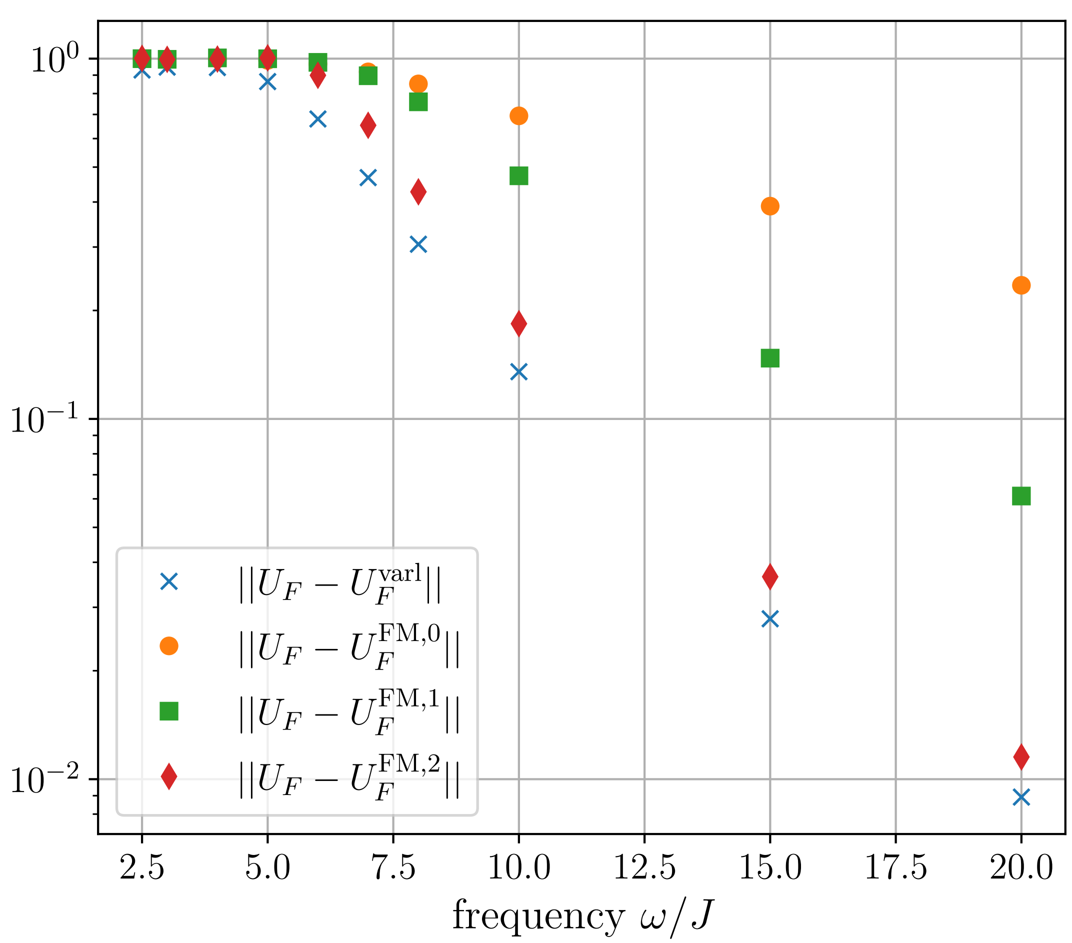
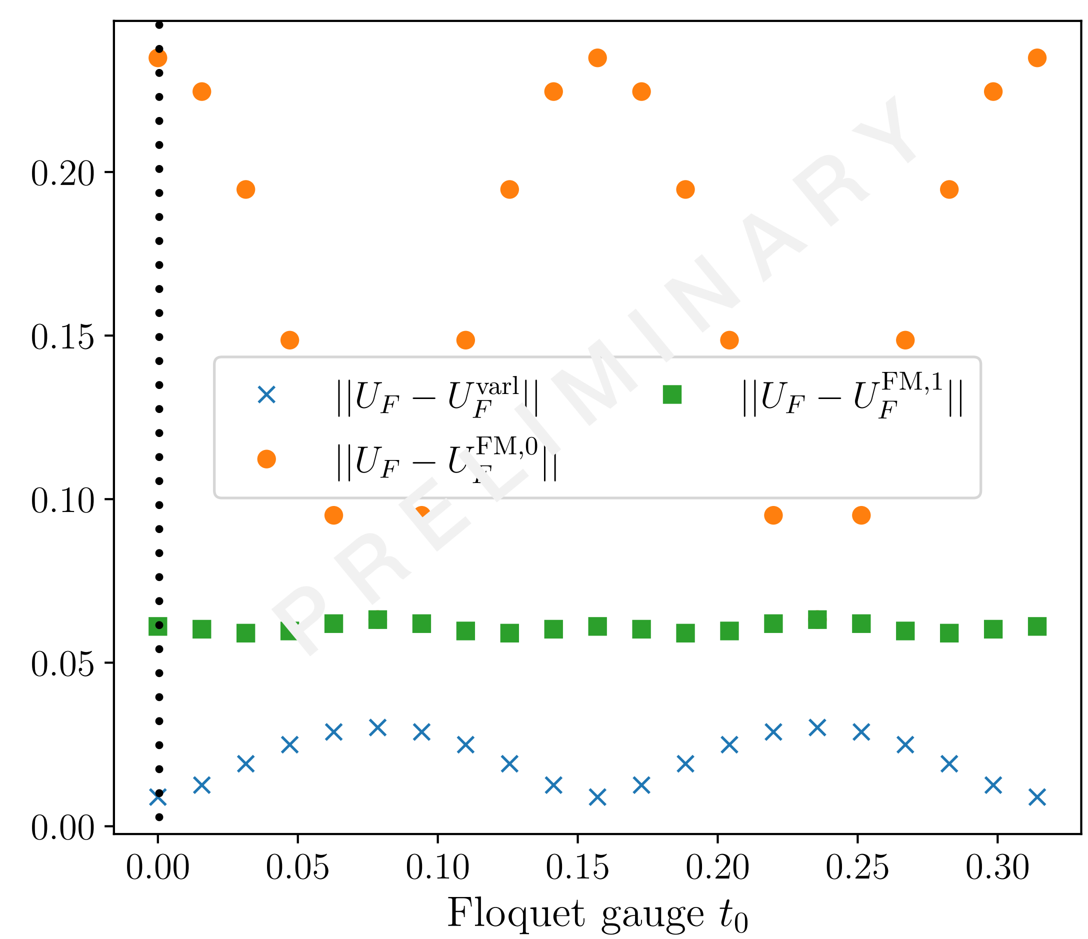
frequency scan



$$t_0 = 0$$

high frequency regime



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Marín Bukov

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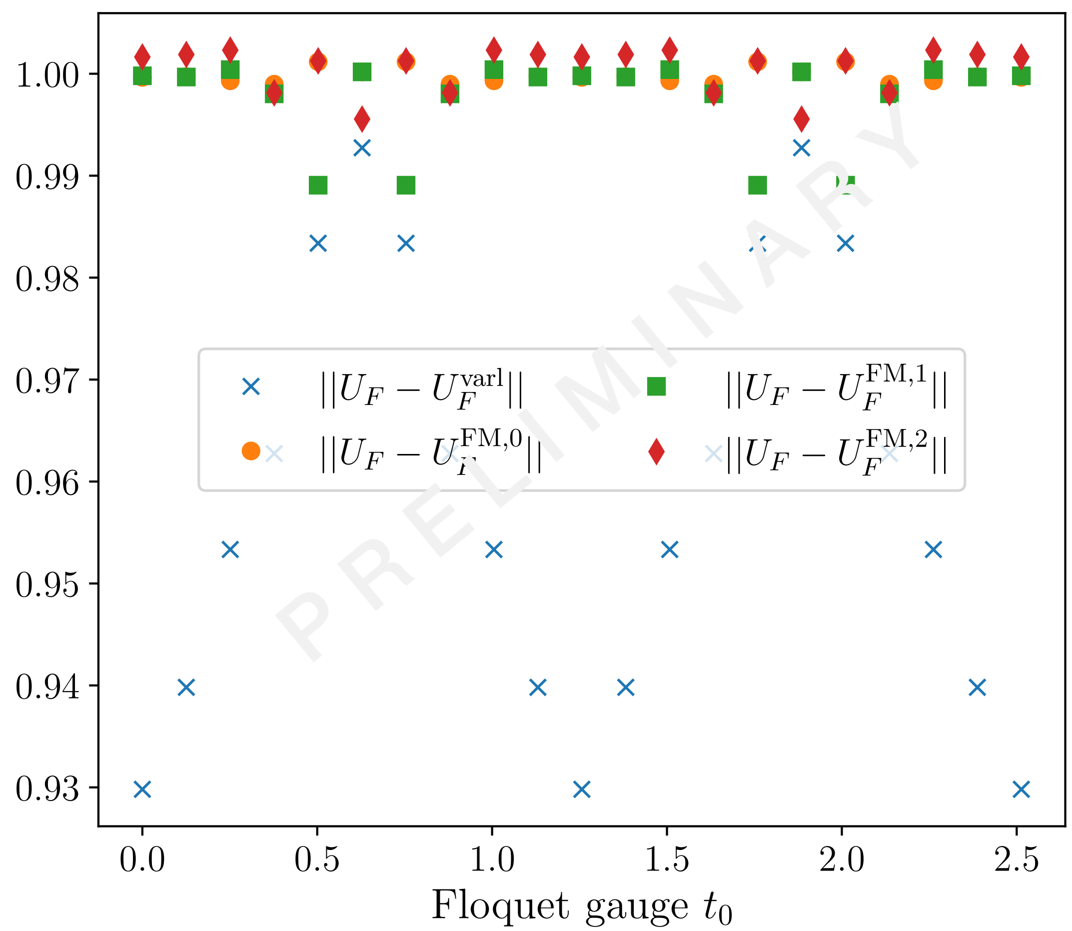
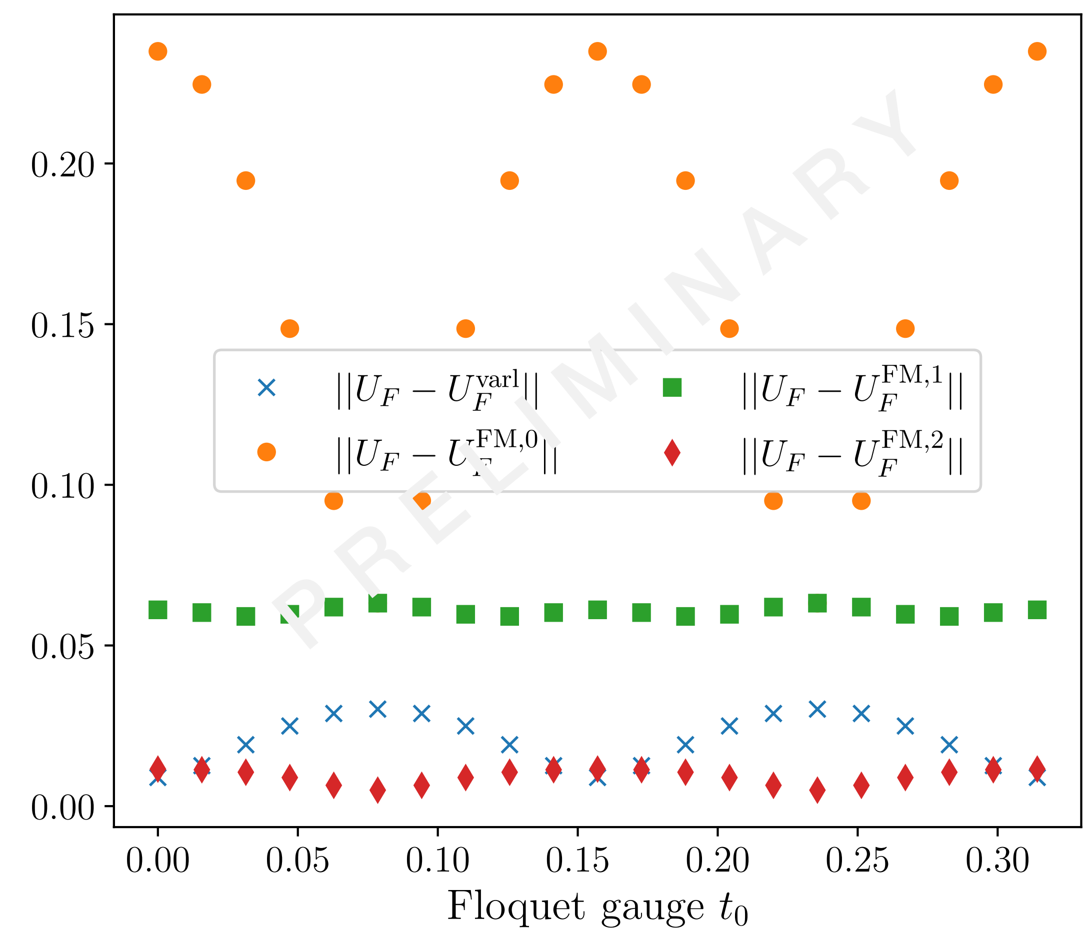
low frequency regime



$\omega/J = 2.5, \quad h_x/J = \sqrt{3}, \quad h_z/J = 0.9$

✓ better than Floquet-Magnus expansion

high frequency regime



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12 **spins,** 21 **harmonics**

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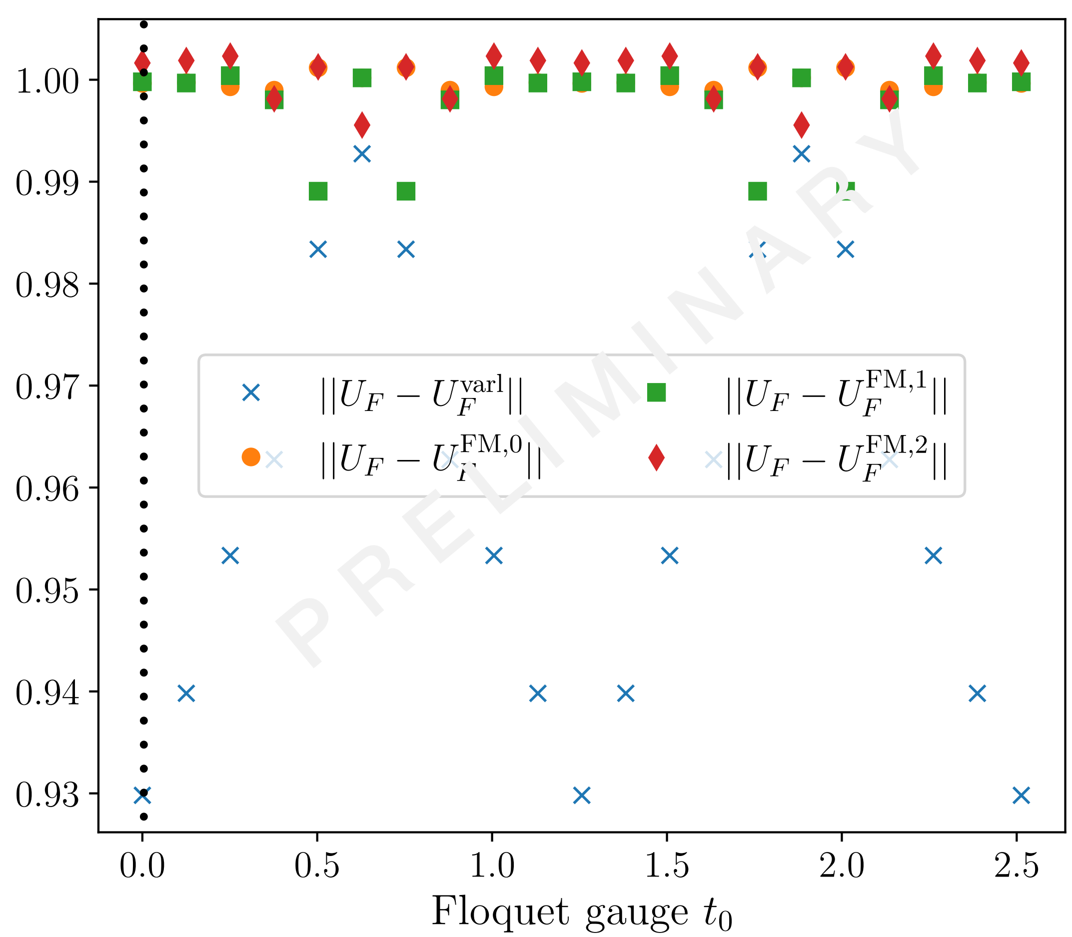

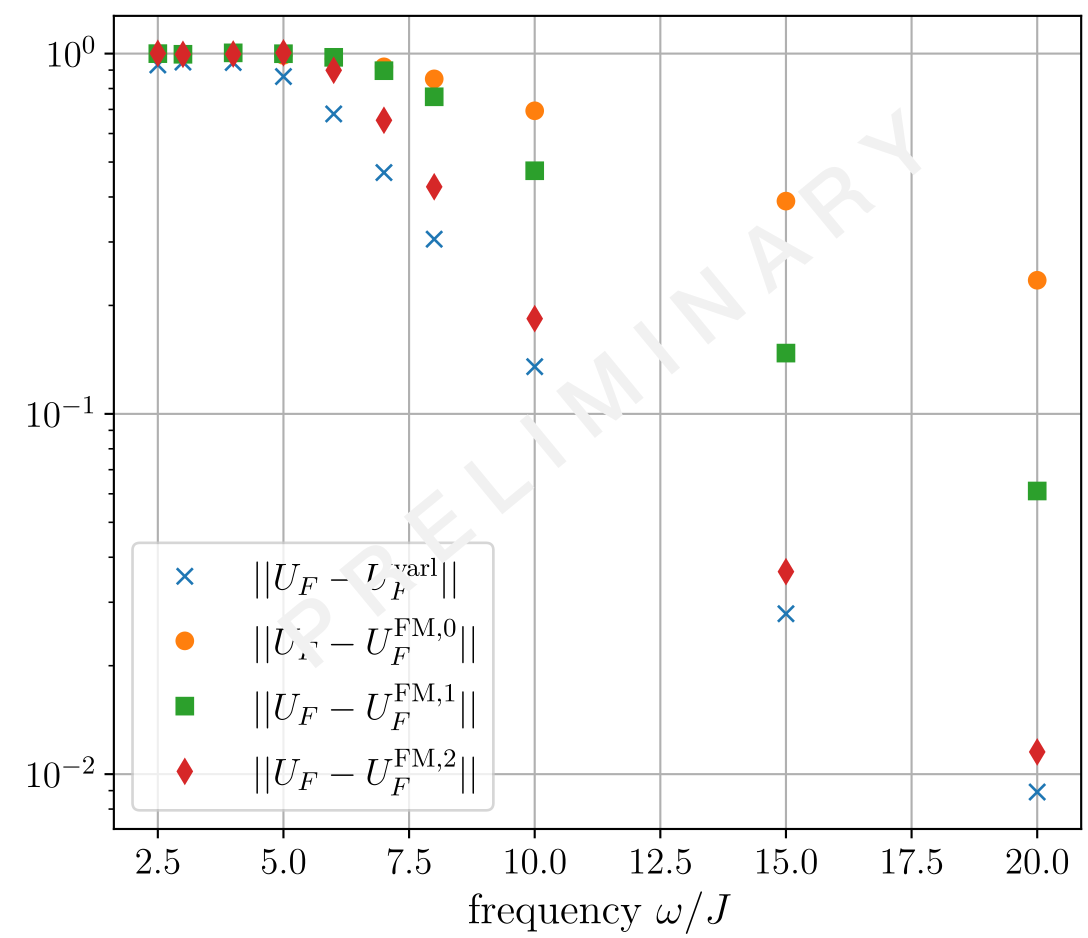

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