

# TOWARDS A CONFORMAL FIELD THEORY FOR CRITICAL PLANAR INTERFACES

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**INHOMOGENEOUS RANDOM SYSTEMS, JANUARY 2026 @ IHP**



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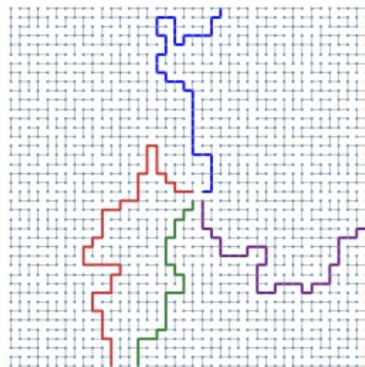
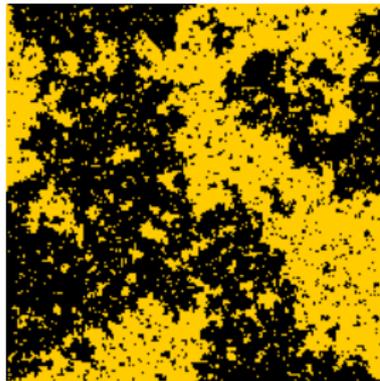
**Aalto University**, Department of Mathematics and Systems Analysis,  
**University of Cologne**, Quantum Matter and Information

INHOMOGENEOUS RANDOM SYSTEMS, JANUARY 2026 @ IHP



## Lattice models in 2D statistical physics

- ▶ phase transitions, **scaling limits**
- ▶ interfaces: **random interacting curves**
- ▶ **conformal invariance** at criticality (?)

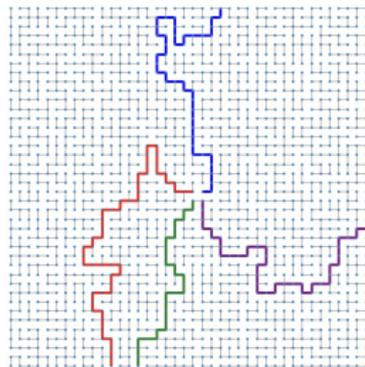
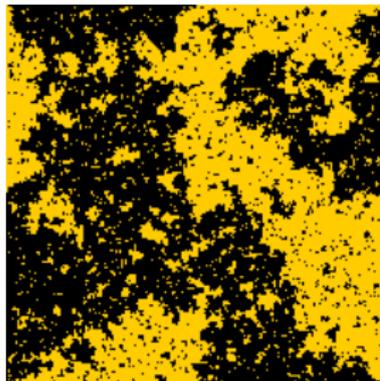


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## Conformal field theory (CFT)

- ▶ operator algebra, **fusion rules**, central charge  $c$
- ▶ singular vectors and null fields
- ⇒ **BPZ PDEs** for correlation functions



## Lattice models in 2D statistical physics

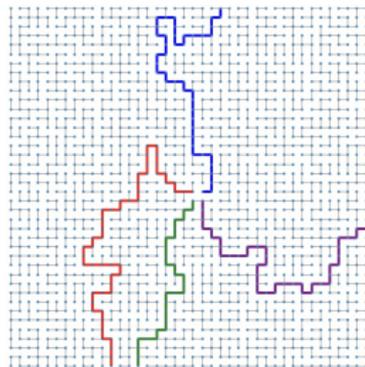
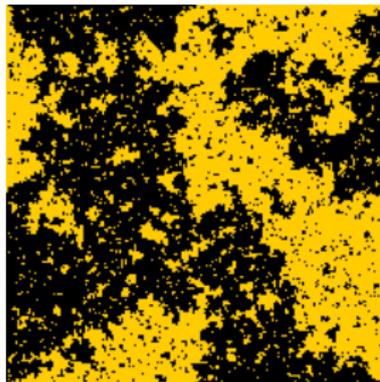
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## Interaction of interfaces

- ▶ probabilities of **topological events**
- ▶ interface “**partition functions**”
- ▶ relation to **correlation functions** in CFT / QFT

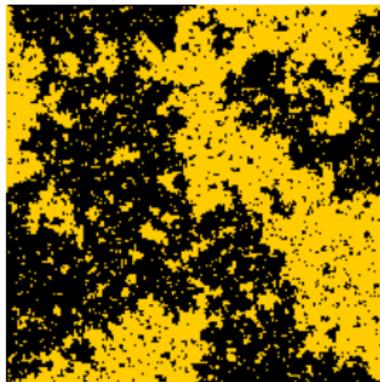


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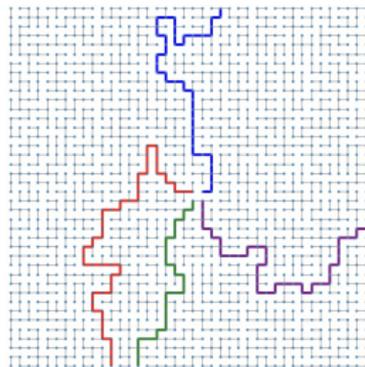


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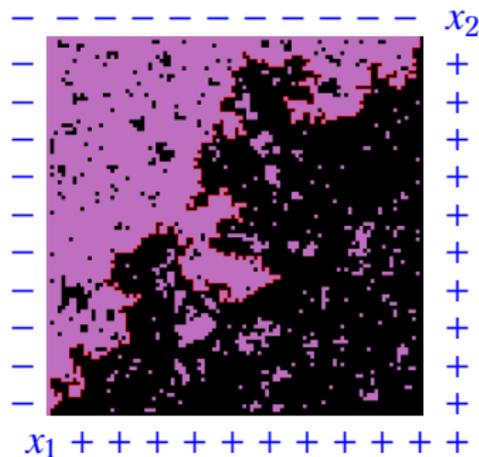
## Various prototypical models...

- |   |               |
|---|---------------|
| ▶ critical <b>percolation</b> (trivial? logarithmic?) | $0$           |
| ▶ critical <b>Ising model</b> (free fermion?)         | $\frac{1}{2}$ |
| ▶ <b>Gaussian free field</b> (free boson?)            | $1?$          |
| ▶ <b>uniform spanning trees</b> (symp. fermion?)      | $-2$          |



# CONFORMAL INVARIANCE IN TERMS OF OBSERVABLES

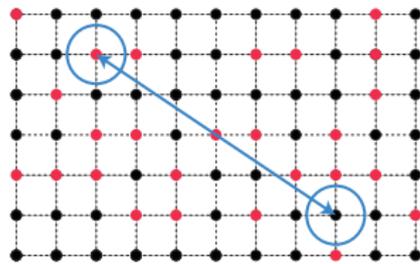
**Interfaces:** SLE( $\kappa$ ) curves



Scaling limit  $\delta \rightarrow 0$   
at criticality ( $T = T_c$ )

$\Rightarrow$  **conformal invariance?**

$$\mathbb{E}[\sigma_{x_1} \sigma_{x_2}]$$

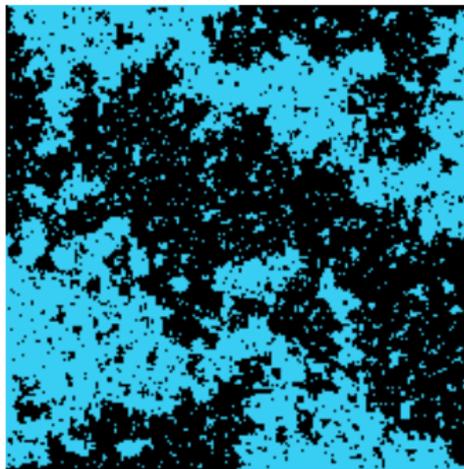
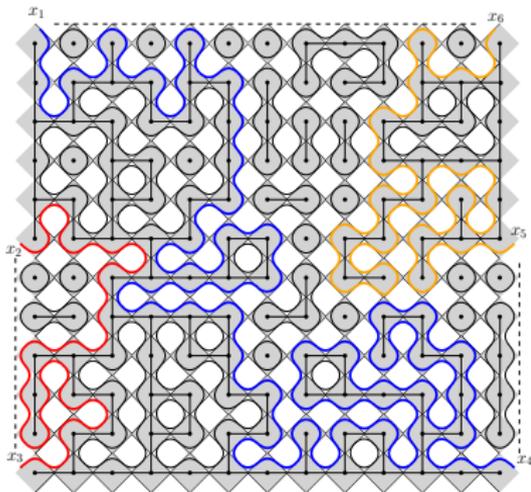


**Correlations** (e.g. between spins)



**Probabilities** of topological events

# PROTOTYPICAL EXAMPLES: PERCOLATION / SPIN MODELS



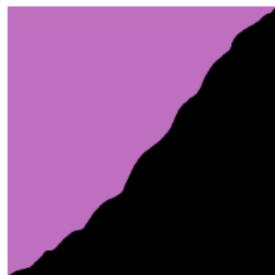
# ISING MODEL: FERROMAGNETIC PHASE TRANSITION

[Lenz & Ising '20s, Peierls 30's, Kramers & Wannier 40's, Onsager 40's →]

- ▶ random **spins**  $\sigma_x = \pm 1$  at vertices  $x$  of a graph
- ▶ nearest neighbor interaction:  $\mathbb{P}[\text{config.}] \propto \exp\left(\frac{1}{T} \sum_{x \sim y} \sigma_x \sigma_y\right)$
- ▶ **phase transition** at **critical** temperature  $T = T_c$

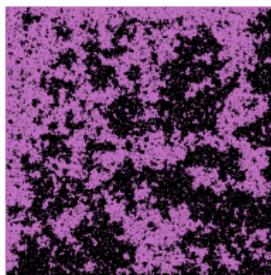
look at *correlation* of a pair of spins at  $x$  and  $y$

$$C(x, y) = \mathbb{E}[\sigma_x \sigma_y] - \mathbb{E}[\sigma_x] \mathbb{E}[\sigma_y] \quad \text{when } |x - y| \gg 1:$$



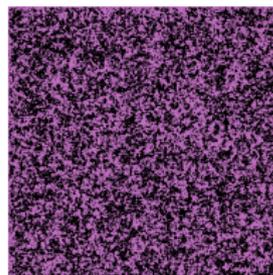
$$T < T_c$$

$$C(x, y) \sim \text{const.}$$



$$T = T_c$$

$$C(x, y) \sim |x - y|^{-\beta}$$



$$T_c < T$$

$$C(x, y) \sim e^{-\frac{1}{\xi}|x-y|}$$

- ▶ scaling limit at critical temperature  $T_c$ : **conformal invariance**?

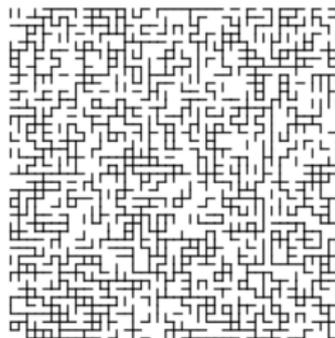
# RANDOM-CLUSTER MODEL (RCM): PERCOLATION

[Fortuin & Kasteleyn '70s, Kesten '80s, ... ]

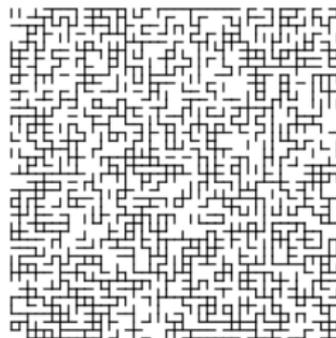
- ▶ random **bonds** (open/closed): configuration  $(\omega(e))_{e \in E} \in \{0, 1\}^E$
- ▶ e.g. Bernoulli bond percolation:  $\omega(e) = 1$  (open) w. proba.  $p \in [0, 1]$  and  $\omega(e) = 0$  (closed) w. proba.  $1 - p$ , all *independently*
- ▶ **RCM**: add *interaction*, namely **cluster-weight**  $q > 0$  and set

$$\mathbb{P}[\omega] \propto p^{\#\text{open in } \omega} (1-p)^{\#\text{closed in } \omega} q^{\#\text{clusters in } \omega}$$

- ▶ unifies percolation, Ising & Potts models, electrical networks, ...



$p = 0.49$



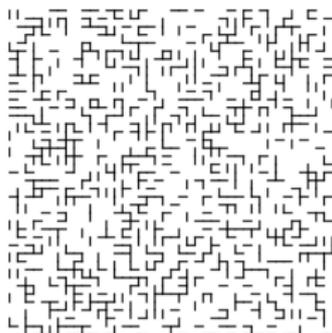
$p = 0.51$

# RANDOM-CLUSTER MODEL – PERCOLATION PHASE TRANSITION

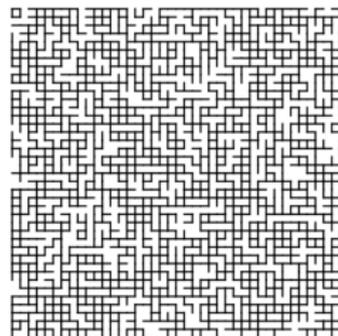
- ▶ **planar RCM**

$$\mathbb{P}[\omega] \propto p^{\#\text{open in } \omega} (1-p)^{\#\text{closed in } \omega} q^{\#\text{clusters in } \omega}$$

- ▶ phase transition in terms of *existence of infinite cluster*



$p = 0.30$



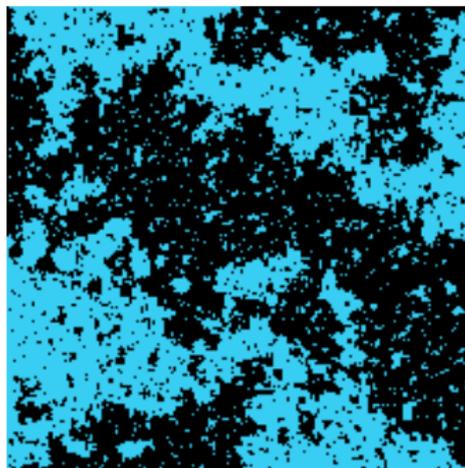
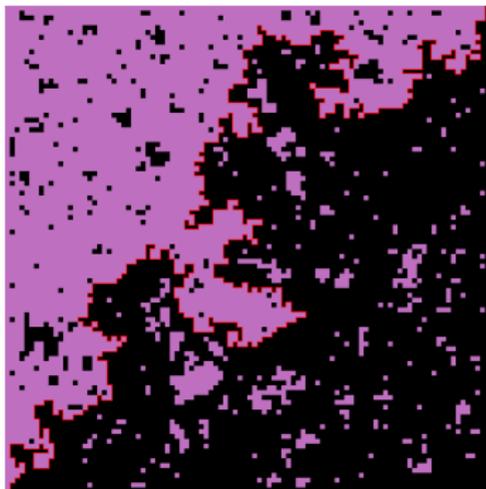
$p = 0.70$

[Grimmett: The RCM book]

- ▶ identified via **self-duality**: when  $q \in [1, 4]$  (expect  $q \in [0, 4]$ ), **continuous phase transition** at critical  $p = p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$

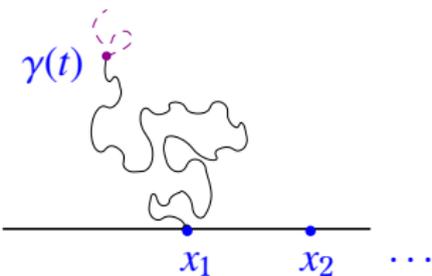
[Baxter '80s, Kesten '80s, ..., Duminil-Copin, Sidoravicius & Tassion '17]

## SCALING LIMITS OF CRITICAL INTERFACES



RANDOM CURVES IN 2D BUILT FROM 1D  
BROWNIAN MOTION

# GROWING SLE( $\kappa$ ) CURVES VIA BROWNIAN MOTION



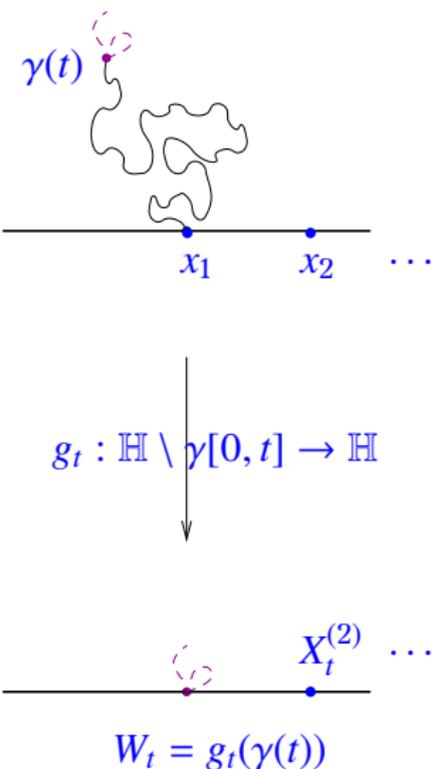
- ▶ *driving process* of one curve  $\gamma$ : image of tip
- ▶ *randomness* from 1D **Brownian motion**  $B$
- ▶ *interaction* encoded in **partition function**  $\mathcal{Z}$ 
  - ▶  $dW_t = \sqrt{\kappa} dB_t + \kappa \partial_1 \log \mathcal{Z}(W_t, X_t^{(2)}, X_t^{(3)}, \dots) dt$
  - ▶  $dX_t^{(i)} = \frac{2dt}{X_t^{(i)} - W_t}$

$$g_t : \mathbb{H} \setminus \gamma[0, t] \rightarrow \mathbb{H}$$



$$W_t = g_t(\gamma(t))$$

# GROWING SLE( $\kappa$ ) CURVES VIA BROWNIAN MOTION



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  - ▶  $dX_t^{(i)} = \frac{2dt}{X_t^{(i)} - W_t}$
- ▶ defining properties:

$\mathcal{Z}$  must satisfy a **BPZ PDE** at the growth point:

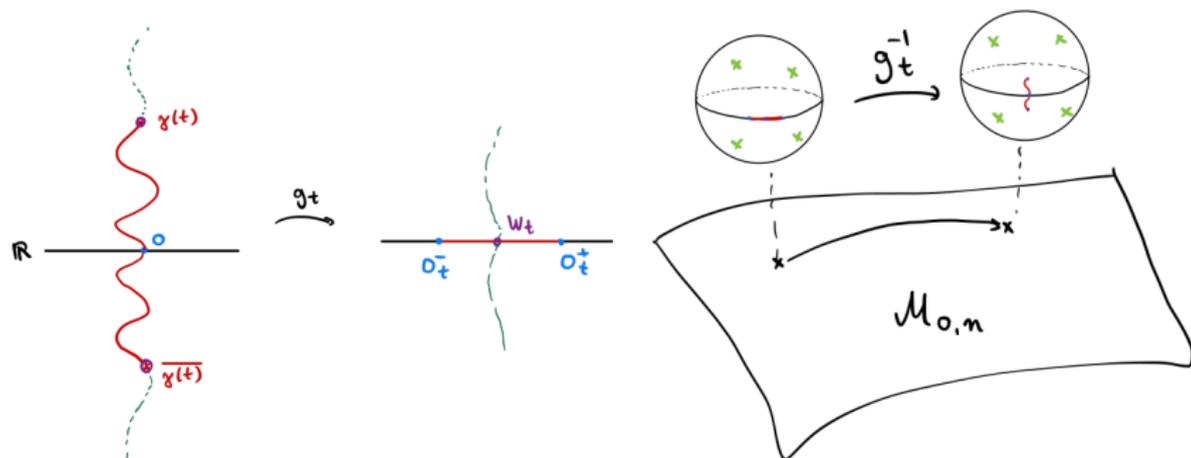
$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_1^2} + \sum_{j \geq 2} \left( \frac{2}{x_j - x_1} \frac{\partial}{\partial x_j} - \frac{2\Delta_j(\kappa)}{(x_j - x_1)^2} \right) \right\} \mathcal{Z}(x_1, \dots, x_n) = 0$$

and global **Möbius covariance**:

$$\mathcal{Z}(f(x_1), \dots, f(x_n)) = \left( \prod_{1 \leq j \leq n} |f'(x_j)|^{\Delta_j(\kappa)} \right) \mathcal{Z}(x_1, \dots, x_n)$$

# IDEA: FOLLOW TIME-EVOLUTION OF THE CURVE

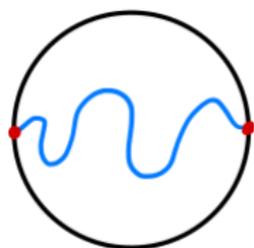
- ▶ curve's **time-evolution** encoded into maps  $g_t: \mathbb{H} \setminus \gamma[0, t] \rightarrow \mathbb{H}$
- ▶ marked points  $z_1, z_2, \dots$  represent **observable locations**
- ▶  $\Delta_1, \Delta_2, \dots \in \mathbb{C}$  are conformal weights / scaling dimensions



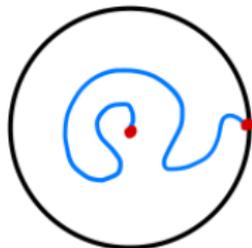
**Martingale observables**  $\implies$  **BPZ PDEs**

$$M_t(x; z_1, \dots) = \left( \prod_j g'_t(z_j)^{\Delta_j} \overline{g'_t(z_j)^{\bar{\Delta}_j}} \right) \mathcal{Z}(W_t; g_t(z_1), \dots)$$

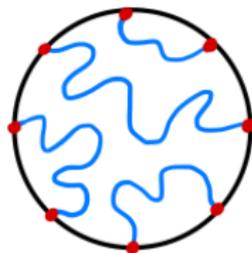
## SLE VARIANTS — INTERACTING CURVES



chordal



radial



multichordal



loop

*Interaction* with underlying domain/surface and other curves encoded in *partition function*  $\mathcal{Z}_D$  ( if finite )

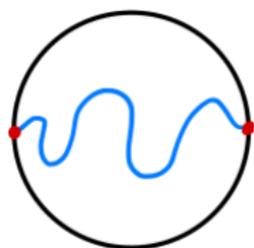
Cardy '03; Bauer, Bernard & Kytölä '05;

Dubédat '06–'07; Kozdron & Lawler '07; Lawler '09;

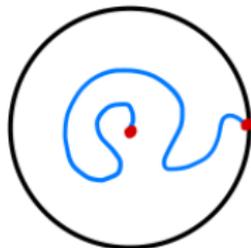
Kytölä & P. '16; Miller & Sheffield '16; P. & Wu '19;

Miller, Sheffield & Werner '20; Beffara, P. & Wu '21, Zhan '24 ...

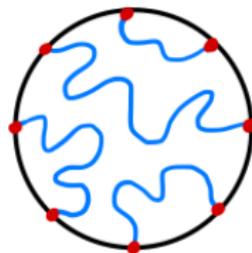
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radial

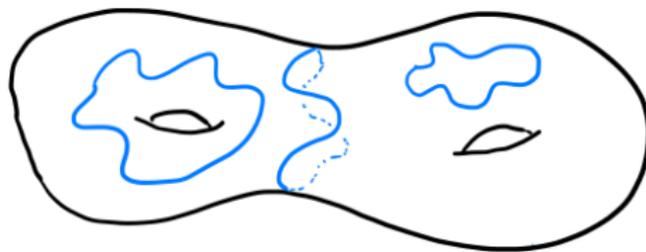


multichordal



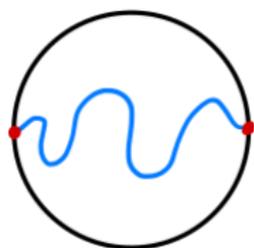
loop

*Interaction* with underlying domain/surface and other curves encoded in *partition function*  $Z_D$  ( if finite )

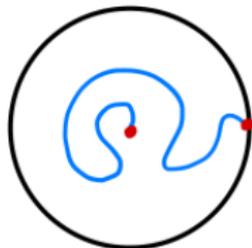


... or various combinations of these...?

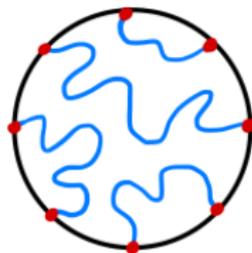
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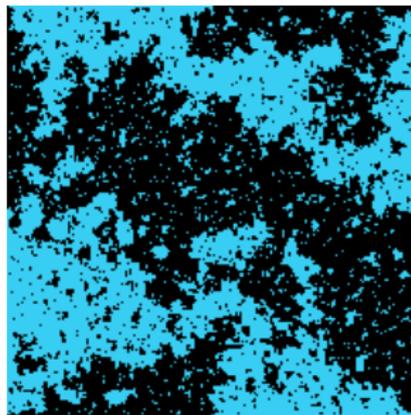
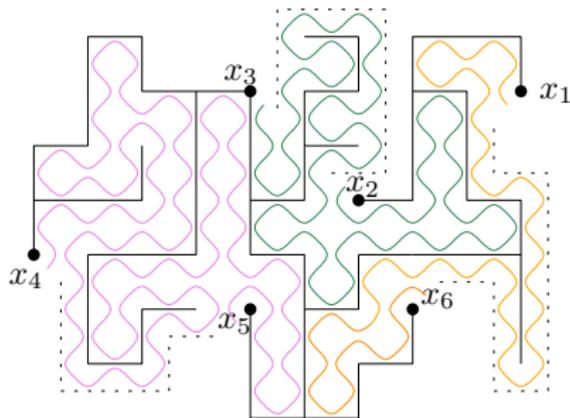
**Interaction with underlying domain/surface and other curves encoded in partition function  $\mathcal{Z}_D$  (if finite)**

$$\frac{d\mu_D}{\bigotimes_{1 \leq j \leq n} d\mu_D^{\text{SLE}(\kappa)}}(\gamma_1, \dots, \gamma_n) := \mathbb{1}\{\gamma_i \cap \gamma_j = \emptyset, i \neq j\} \exp\left(\frac{c(\kappa)}{2} m_D(\gamma_1, \dots, \gamma_n)\right)$$

Normalize to proba measure:  $\mathbb{P}_D := \frac{\mu_D}{\mathcal{Z}_D(x_1, \dots, x_p)}$ , where

$$\mathcal{Z}_D(x_1, \dots, x_p) = \int_{\text{curves w endpoints } x_1, \dots, x_p} \mathbb{1}\{\dots\} \exp\left(\frac{c(\kappa)}{2} m_D(\gamma_1, \dots, \gamma_n)\right) \bigotimes_{1 \leq j \leq n} d\mu_D^{\text{SLE}(\kappa)}(\gamma_1, \dots, \gamma_n)$$

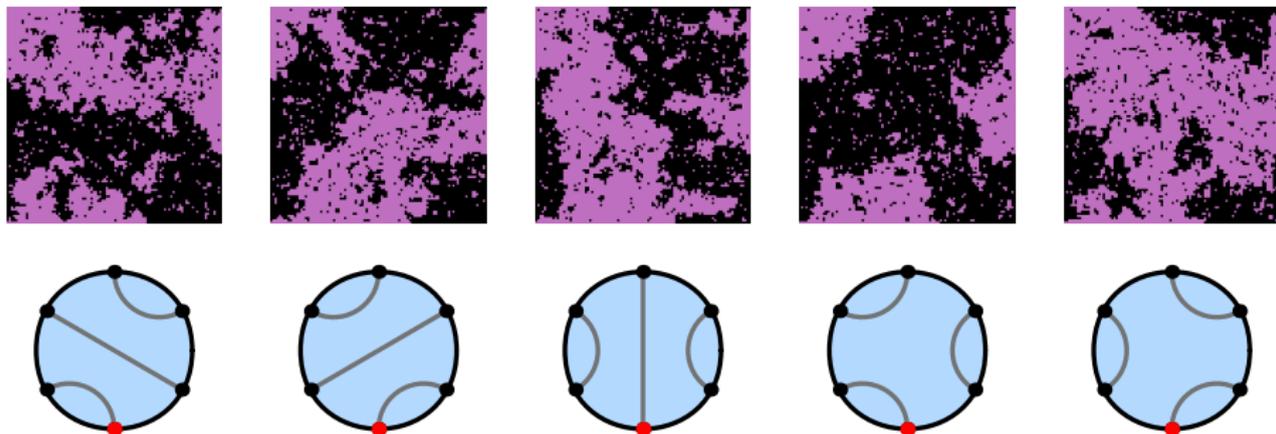
# CROSSING PROBABILITIES GIVEN BY SLE( $\kappa$ ) PURE PARTITION FUNCTIONS



$$dW_t = \sqrt{\kappa} dB_t + \kappa \partial_1 \log \mathcal{Z}(W_t, X_t^{(2)}, X_t^{(3)}, \dots, X_t^{(N)}) dt$$

# CROSSING PROBABILITIES IN CRITICAL 2D ISING MODEL

- ▶ “alternating”  $\oplus/\ominus$  boundary conditions  
 $\implies N$  macroscopic interfaces connect the marked points pairwise
- ▶ possible connectivities labeled by *planar pair partitions*  $\alpha \in \text{LP}_N$



What are the probabilities of the various connectivities?

# CROSSING PROBABILITIES OF CRITICAL ISING INTERFACES

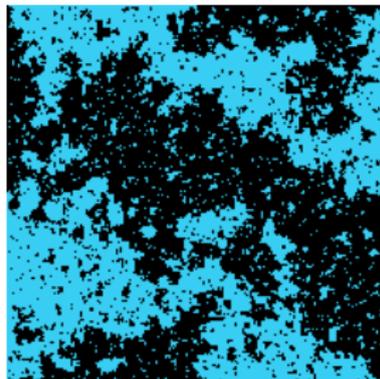
$$(\Omega^\delta; x_1^\delta, \dots, x_{2N}^\delta) \xrightarrow{\delta \rightarrow 0} (\Omega; x_1, \dots, x_{2N}) \quad (\text{in close-Carathéodory sense})$$

**Thm.** [Izyurov '11-'15; P. & Wu '22]

For critical **Ising model** on  $\Omega^\delta$  with alternating boundary cond.,

$$\lim_{\delta \rightarrow 0} \mathbb{P}^\delta [\text{conn.} = \alpha] \propto \mathcal{Z}_\alpha^{(k=3)}(\Omega; x_1, \dots, x_{2N}) \quad \alpha \in \text{LP}_N$$

- ▶  $\{\mathcal{Z}_\alpha^{(k)} : \alpha \in \text{LP}_N\}$  “pure” partition functions  
(interaction & conditioning)
- ▶ **match with  $c = 1/2$  CFT fields  $\phi_{1,2}$** 
  - ▶ conformal invariance
  - ▶ BPZ PDEs
  - ▶ fusion asymptotics
- ▶ **NB: NOT a minimal model !**



predictions: [Cardy '00s, Bauer, Bernard & Kytölä '05]

$$\mathcal{Z}_{\text{arc}}(z) = \frac{2\Gamma(4/3)}{\Gamma(5/3)\Gamma(8/3)} z^{2/3}(1-z)^{-1} {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; z\right)$$

$$\mathcal{Z}_{\text{two arcs}}(z) = \frac{2\Gamma(4/3)}{\Gamma(5/3)\Gamma(8/3)} (1-z)^{2/3} z^{-1} {}_2F_1\left(\frac{4}{\kappa}, 1 - \frac{4}{\kappa}, \frac{8}{\kappa}; 1-z\right)$$

Compare with Ising minimal model CFT:

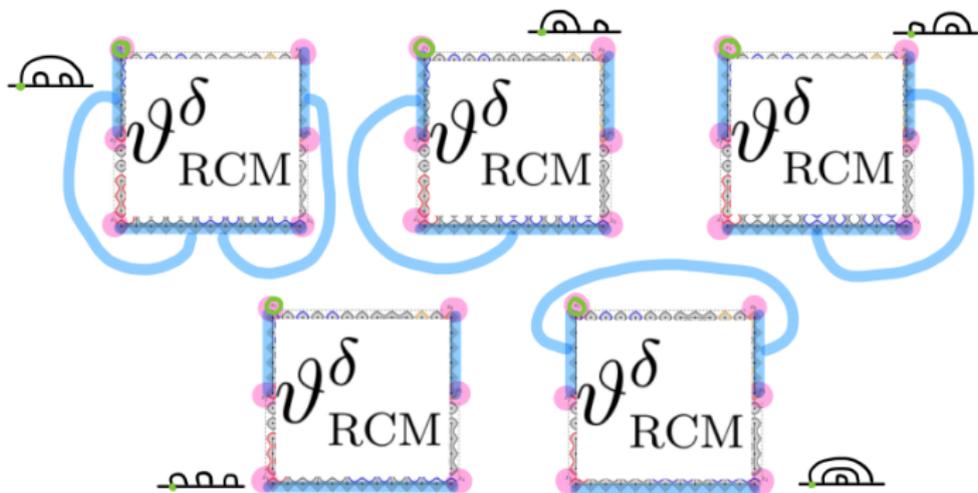
$$\mathcal{Z}_{\text{arc}}(z) + \mathcal{Z}_{\text{two arcs}}(z) = \frac{z^2 - z + 1}{z(1-z)} = z^{2/3}(1-z)^{2/3} I_2(z),$$

- ▶  $I_2(z) = \langle \varepsilon(0) \varepsilon(z) \varepsilon(1) \varepsilon(\infty) \rangle$  is what's in CFT literature known as the minimal model *energy field*  $\varepsilon$  chiral conformal block
- ▶ energy field  $\varepsilon$  has conformal weight / scaling dim.  $h_{1,2} = \frac{1}{2}$
- ▶ here  $\kappa = 3$ , and central charge is  $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa} = \frac{1}{2}$

# BOUNDARY CONDITIONS (B.C.) CAN BE MORE COMPLICATED

- ▶ for RCM, labeled by planar pair partitions  $\beta \in LP_N$
- ▶ combinatorics for them encoded in **meander matrix**:

$$\mathcal{M}_{\alpha,\beta}(q) := q^{\frac{1}{2}\#\text{loops in } (\alpha,\beta)} \quad \text{for } (\alpha,\beta) = \text{---}\overset{\frown}{\text{---}}\text{---}\circ\text{---}$$



- ▶ **internal connectivities**  $\vartheta(\omega) = \vartheta_{RCM}^\delta$  of interfaces labeled  $\alpha \in LP_N$

# CROSSING PROBABILITIES OF CRITICAL FK-ISING INTERFACES

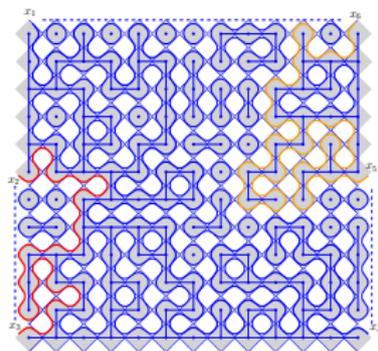
$$(\Omega^\delta; x_1^\delta, \dots, x_{2N}^\delta) \xrightarrow{\delta \rightarrow 0} (\Omega; x_1, \dots, x_{2N}) \quad (\text{in close-Carathéodory sense})$$

**Thm.** [Izyurov '22; Feng, P. & Wu '22]

For critical FK-Ising model on  $\Omega^\delta$  with any b.c.  $\beta \in \text{LP}_N$ ,

$$\lim_{\delta \rightarrow 0} \mathbb{P}_\beta^\delta [\text{conn.} = \alpha] = \mathcal{M}_{\alpha, \beta}^{(\kappa=16/3)} \frac{\mathcal{Z}_\alpha^{(\kappa=16/3)}(\Omega; x_1, \dots, x_{2N})}{\mathcal{F}_\beta^{(\kappa=16/3)}(\Omega; x_1, \dots, x_{2N})}, \quad \alpha \in \text{LP}_N$$

- ▶  $\{\mathcal{Z}_\alpha^{(\kappa)} : \alpha \in \text{LP}_N\}$  “pure” partition functions
- ▶  $\{\mathcal{F}_\beta^{(\kappa)} : \beta \in \text{LP}_N\}$  partition functions for b.c.
- ▶  $\mathcal{M}^{(\kappa)}$ : meander matrix (from CFT algebraic content:  $U_q(\mathfrak{sl}_2)$  & TL algebra action)
- ▶ match with  $c = 1/2$  CFT fields  $\phi_{1,2}$
- ▶ **NB:** NOT a minimal model !



predictions: [Flores, Kleban, Simmons & Ziff '17]

# CROSSING PROBABILITIES OF CRITICAL RCM INTERFACES

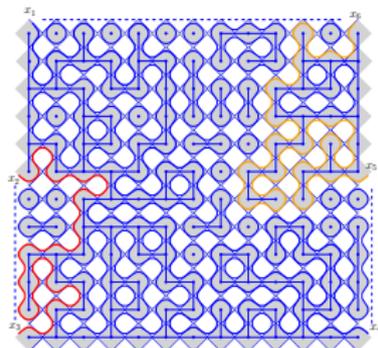
$$(\Omega^\delta; x_1^\delta, \dots, x_{2N}^\delta) \xrightarrow{\delta \rightarrow 0} (\Omega; x_1, \dots, x_{2N}) \quad (\text{in close-Carathéodory sense})$$

**Conjecture.** ( for  $q(\kappa) \in [0, 4]$  i.e.  $\kappa \in [4, 8]$  – proven for  $\kappa = (4), (\frac{16}{3}), (6), 8$  )

For critical **random-cluster model** on  $\Omega^\delta$  with any b.c.  $\beta \in \text{LP}_N$ ,

$$\lim_{\delta \rightarrow 0} \mathbb{P}_\beta^\delta [\text{conn.} = \alpha] = \mathcal{M}_{\alpha, \beta}^{(\kappa)} \frac{\mathcal{Z}_\alpha^{(\kappa)}(\Omega; x_1, \dots, x_{2N})}{\mathcal{F}_\beta^{(\kappa)}(\Omega; x_1, \dots, x_{2N})}, \quad \alpha \in \text{LP}_N$$

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- ▶  $\{\mathcal{F}_\beta^{(\kappa)} : \beta \in \text{LP}_N\}$  partition functions for b.c.
- ▶  $\mathcal{M}^{(\kappa)}$ : meander matrix
- ▶ conformal invariance, BPZ PDEs, fusion  
 $c = \frac{1}{2\kappa}(6 - \kappa)(3\kappa - 8)$  CFT fields  $\phi_{1,2}$
- ▶ **NB:** NOT a minimal model: **log-CFT** ?



predictions: [Flores, Kleban, Simmons & Ziff '17]

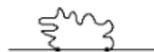
# LOGARITHMIC FUSION FOR SLE(8) BOUNDARY FIELD $\phi_{1,2}$

Thm. [Liu & P. & Wu '21]

Explicit fusion rules from SLE(8) pure partition functions  $\mathcal{Z}_\alpha^{(\kappa=8)}$ :

- ▶ “zero-leg channel”

$$\frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{|x_{j+1} - x_j|^{1/4} \log |x_{j+1} - x_j|}$$



$$\xrightarrow{x_j, x_{j+1} \rightarrow \xi} \mathcal{Z}_{\alpha \setminus \{j, j+1\}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) \quad \text{if } \{j, j+1\} \in \alpha$$

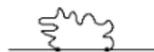
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$$\begin{matrix} x_j, x_{j+1} \rightarrow \xi \\ \longrightarrow \end{matrix} \mathcal{Z}_{\alpha \setminus \{j, j+1\}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) \quad \text{if } \{j, j+1\} \in \alpha$$

- ▶ “two-leg channel” (NB: limit is independent of  $\xi$ )

$$\frac{\mathcal{Z}_\alpha(x_1, \dots, x_{2N})}{|x_{j+1} - x_j|^{1/4}}$$



$$\begin{matrix} x_j, x_{j+1} \rightarrow \xi \\ \longrightarrow \end{matrix} \pi \mathcal{Z}_{\varphi(\alpha) \setminus \{j, j+1\}}(x_1, \dots, x_{j-1}, x_{j+2}, \dots, x_{2N}) \quad \text{if } \{j, j+1\} \notin \alpha$$

# LOGARITHMIC FUSION FOR $SLE(8)$ BOUNDARY FIELD $\phi_{1,2}$

## Consequence.

For *any* CFT boundary fields describing  $SLE(8)$  curves,  
(whatever that'd mean...)

OPE product has *explicit form*

$$\phi_{1,2}(z) \phi_{1,2}(w) \sim (z-w)^{-1/4} \left( \pi \phi_{1,1}(z) - \log(z-w) \widetilde{\phi}_{1,3}(z) \right).$$

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Compare with fusion of two simple Virasoro modules with  $c = -2$ :

- ▶ for simple module  $S_{1,2}$  (corresponding to  $\phi_{1,2}$  with  $\kappa = 8$ ):

$$0 \longrightarrow S_{1,1} \xrightarrow{\iota} S_{1,2} \boxtimes S_{1,2} \xrightarrow{\pi} S_{1,3} \longrightarrow 0$$

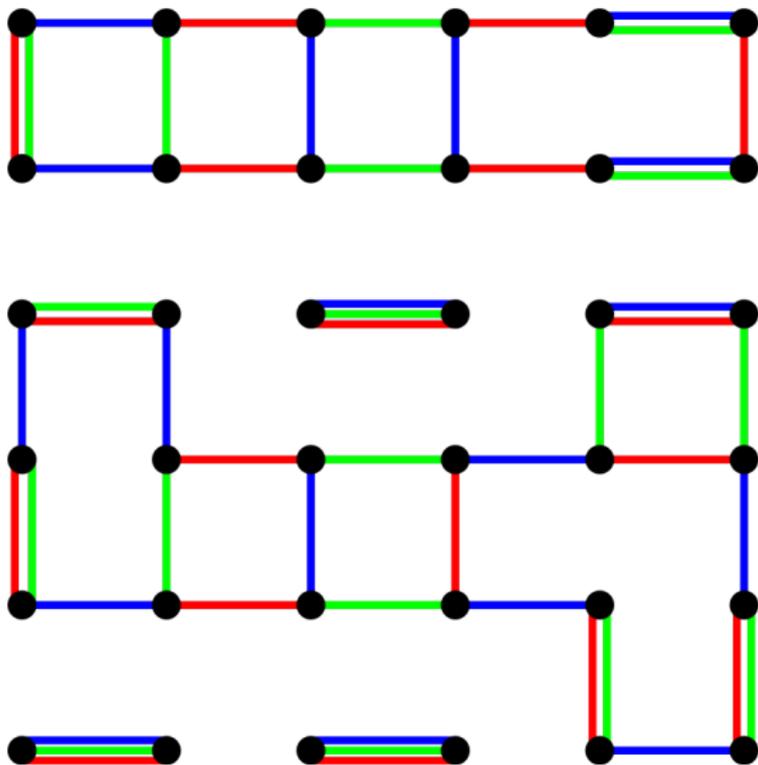
where  $S_{1,2} \boxtimes S_{1,2}$  is so-called *staggered module* (not semisimple)

- ▶  $S_{1,1}$  corresponding to  $\phi_{1,1}$
- ▶  $S_{1,3}$  corresponding to its “log-partner”  $\tilde{\phi}_{1,3}$

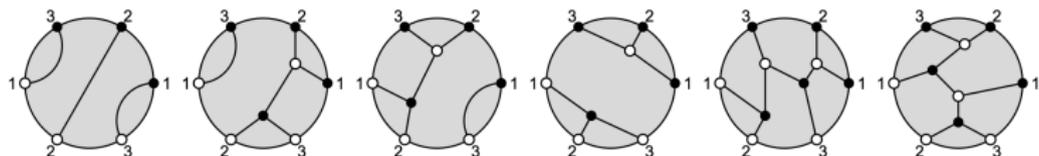
[Gurarie '93; Gaberdiel & Kausch '96; Rohsiepe '96; Kytölä & Ridout '09]



# TRIPLE DIMERS ( $c = 2$ ) AND $W_3$ -CONFORMAL BLOCKS?



# BEYOND CURVES?



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Thm. [Lafay, Le & Roussillon 2025]

$$\mathbb{P}_\beta^\delta[\text{triple dimer web} = \alpha] \xrightarrow{\delta \rightarrow 0} \mathcal{K}_{\alpha,\beta} \frac{\mathcal{Z}_\alpha(x_1, \dots, x_d)}{\mathcal{U}_\beta(x_1, \dots, x_d)}$$

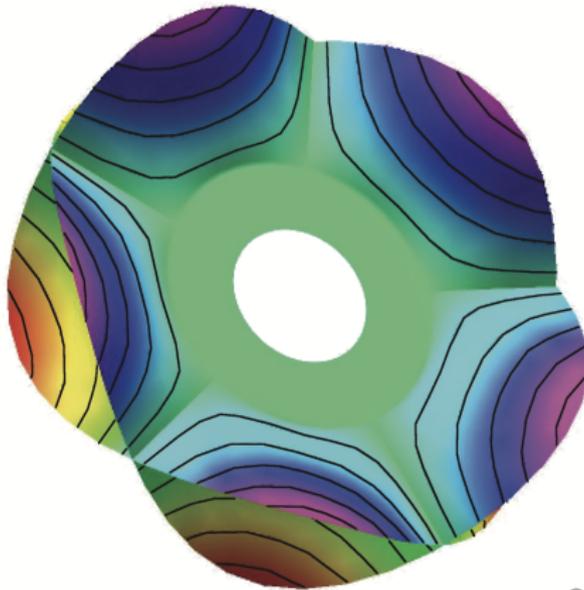
- ▶ where  $\mathcal{U}_\beta$  is the conformal block function (fused Specht polynomial)

$$\mathcal{U}_\beta(x_1, \dots, x_d) = \prod_{i < j} (x_j - x_i)^{-s_i s_j / 3} \mathcal{P}_{\beta^r}(x_1, \dots, x_d)$$

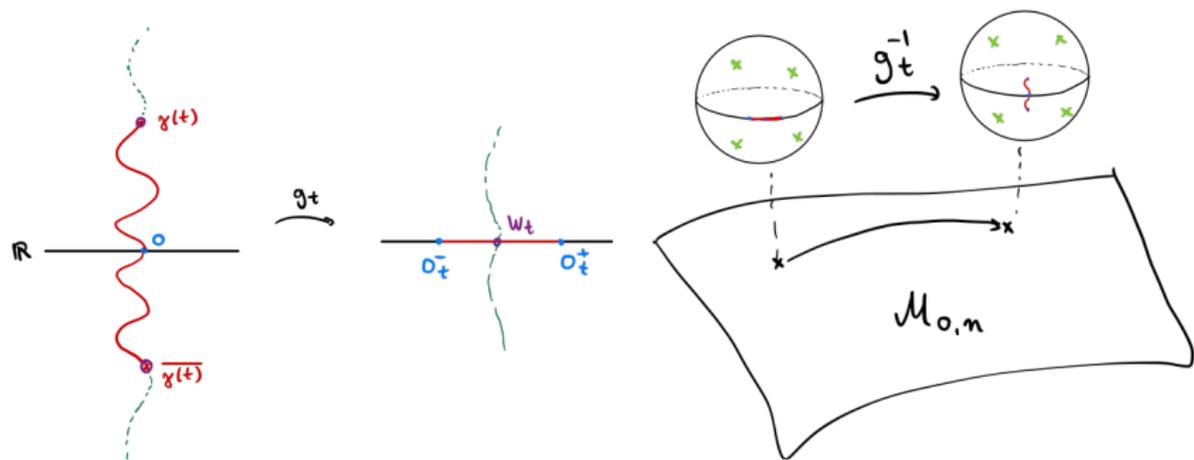
- ▶  $\mathcal{Z}_\alpha := \sum_\beta \mathcal{K}_{\alpha,\beta}^{-1} \mathcal{U}_\beta$  and  $\mathcal{K}_{\alpha,\beta}, \mathcal{K}_{\alpha,\beta}^{-1}$  combinatorial numbers

**Upshot:** Triple-dimer connection probabilities are given by specific CFT correlation functions with  $W_3$  algebra symmetry!

# BEYOND REGULAR SINGULARITIES?



# BYOND REGULAR SINGULARITIES?



$$M_t(x; z_1, \dots) = \left( \prod_j g'_t(z_j)^{\Delta_j} \overline{g'_t(z_j)^{\bar{\Delta}_j}} \right) \mathcal{Z}(W_t; g_t(z_1), \dots)$$

# PARTITION FUNCTION WITH IRREGULAR SINGULARITIES: $\kappa = 4$

Thm. [Desiraju, Korzhenkova & P. 2025+]

We find a matrix-valued (local) martingale for the SLE(4) curve:

$$M_t = \left( \prod_{i=1}^n g'_t(z_i)^{\Delta_i} \exp \left( \frac{s_i^2}{6} \mathcal{S}(g_t)(z_i) + s_i \sqrt{\Delta_i} \mathcal{A}(g_t)(z_i) \right) \right) \mathcal{Z}(W_t, g_t(\mathbf{z}); g_t(\mathbf{s}))$$

- ▶  $W_t = 2B_t \in \mathbb{R}$ : Loewner driving function of the SLE(4) curve,
- ▶  $g_t(z_i)$  time-evol. of punctures  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$
- ▶  $g_t(s_i)$  time-evol. of *Birkhoff spectral invariants*  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$   
(here, eigenvalues of matrices  $A_{i,1}$  in non-Fuchsian system)
- ▶  $\mathcal{A}(g) := g''/g'$  is the pre-Schwarzian
- ▶  $\mathcal{S}(g) := (g''/g')' - \frac{1}{2}(g''/g')^2$  is the Schwarzian
- ▶  $\mathcal{Z}(W_t, g_t(\mathbf{z}); g_t(\mathbf{s})) = \tau(g_t(\mathbf{z}); g_t(\mathbf{s})) Y(W_t, g_t(\mathbf{z}); g_t(\mathbf{s}))$ ,  
where  $Y(x, \mathbf{z}; \mathbf{s}) \sim (x - z_i)^{-2\Delta_i} \exp \left( -\frac{s_i}{x - z_i} \right)$

# PARTITION FUNCTION WITH IRREGULAR SINGULARITIES: $\kappa = 4$

Thm. [Desiraju, Korzhenkova & P. 2025+]

The quantity

$$M_t = \left( \prod_{i=1}^n g'_t(z_i)^{\Delta_i} \exp \left( \frac{s_i^2}{6} \mathcal{S}(g_t)(z_i) + s_i \sqrt{\Delta_i} \mathcal{A}(g_t)(z_i) \right) \right) \mathcal{Z}(W_t, g_t(\mathbf{z}); g_t(\mathbf{s})),$$

is a (local) martingale iff  $\mathcal{Z}$  solves the **confluent BPZ PDEs**

$$\left[ 2 \frac{\partial^2}{\partial x^2} - \sum_{i=1}^n \left( \frac{2}{x - z_i} \frac{\partial}{\partial z_i} + \frac{2s_i}{(x - z_i)^2} \frac{\partial}{\partial s_i} \right. \right. \\ \left. \left. + \frac{2\Delta_i}{(x - z_i)^2} + \frac{4s_i \sqrt{\Delta_i}}{(x - z_i)^3} + \frac{2s_i^2}{(x - z_i)^4} \right) \right] \mathcal{Z}(x, \mathbf{z}; \mathbf{s}) = 0$$

## UNDERLYING (NON-FUCHSIAN) SYSTEM

If someone is familiar... here it is:

$$\frac{\partial}{\partial x} Y(x; \mathbf{z}, \mathbf{s}) = A(x, \mathbf{z}) Y(x; \mathbf{z}; \mathbf{s})$$

$$A(x, \mathbf{z}) := \sum_{i=1}^n \frac{A_{i,0}(\mathbf{z})}{x - z_i} + \frac{A_{i,1}(\mathbf{z})}{(x - z_i)^2}, \quad A_{i,j}(\mathbf{z}) \in \mathfrak{sl}_2(\mathbb{C})$$

- ▶  $\mathbf{z} = (z_1, \dots, z_n)$  and  $\mathbf{s} = (s_1, \dots, s_n)$
- ▶  $Y(x, \mathbf{z}; \mathbf{s})$  is the local solution in coordinate  $x$
- ▶  $\tau(\mathbf{z}; \mathbf{s})$ : tau-function of the related integrable system, defined via

$$(\partial_{\bullet} \log \tau(\mathbf{z}, \mathbf{s})) d\bullet := H_{\bullet}, \quad \bullet = \{s_i, z_i\},$$

where the Hamiltonians are  $\det(A(y, \mathbf{z}) - \sigma(y, \mathbf{z}, \mathbf{s})\mathbf{1}) = 0$  and

$$H_{z_i} := \frac{1}{2} \operatorname{res}_{y=z_i} \operatorname{Tr}(A(y, \mathbf{z}))^2 dz, \quad H_{s_i} := -2 \operatorname{res}_{y=z_i} \frac{\sigma(y, \mathbf{z}, \mathbf{s})}{(y - z_i)} dz$$

# PARTITION FUNCTION WITH IRREGULAR SINGULARITIES: $\kappa = 4$

Puzzle. What is the interpretation of the observable

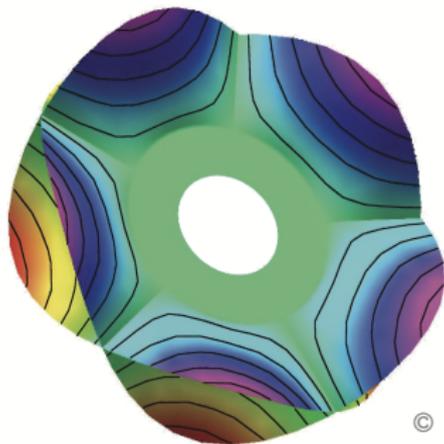
$$\left( \prod_{i=1}^n g'_t(z_i)^{\Delta_i} e^{\frac{s_i^2}{6} \mathcal{S}(g_t)(z_i) + s_i \sqrt{\Delta_i} \mathcal{A}(g_t)(z_i)} \tau(g_t(\mathbf{z}); g_t(\mathbf{s})) Y(W_t, g_t(\mathbf{z}); g_t(\mathbf{s})) \right) ???$$

- ▶ dependence of *curve tip*  $x$  is in  $Y(x, \mathbf{z}; \mathbf{s}) \sim (x - z_i)^{-2\Delta_i} \exp\left(-\frac{s_i}{x - z_i}\right)$
- ▶  $\Delta_i$ : monodromy when SLE curve winds around puncture  $z_i$

[Dubédat '19, Chelkak & Basok '21]

- ▶ ... but the irregular nature of the singularities also introduces

*Stokes phenomenon...*



THANKS !

