From shocks to stochasticity: modern perspectives on Burgers turbulence

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Hydrodynamic turbulence

Incompressible Navier Stokes equation $\boldsymbol{x} \in \mathbb{T}^d$, $t > t_0$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0$$
$$\boldsymbol{u}(\boldsymbol{x}, t_0) = \boldsymbol{u}_0(\boldsymbol{x})$$

Random forcing $\boldsymbol{f}$ injects energy at large scales, dissipation acts at small scales.

Turbulent invariant measure: $\mu_{\mathrm{turb}}(\mathrm{d}\boldsymbol{u}) = \lim_{\nu \to 0} \lim_{t_0 \to -\infty} \mu_\nu(\mathrm{d}\boldsymbol{u}, t \mid \boldsymbol{u}_0, t_0)$

Relations to Euler dynamics obtained by setting $\nu = 0$?

Three key phenomena:
Dissipative anomaly, spatial intermittency, and spontaneous stochasticity.

Dissipative anomaly

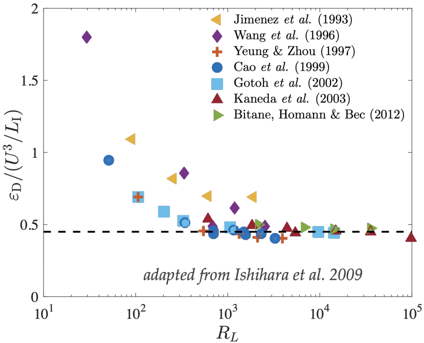
Observable $\langle \mathcal{F} \rangle_\nu := \int \mathcal{F}[\boldsymbol{u}]\, \mathrm{d}\mu_\nu(\mathrm{d}\boldsymbol{u}) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathcal{F} \rangle_\nu + \underbrace{\mathcal{T}}_{\text{nonlinear transfers}} = \underbrace{\mathcal{D}_\nu}_{\text{viscous contribution}} + \underbrace{\mathcal{I}}_{\text{injection}}$

Dissipative anomaly: $\boxed{\lim_{\nu \to 0} \mathcal{D}_\nu \neq 0}$

Case of energy density $E = \frac{1}{2} \left\langle |\boldsymbol{u}|^2 \right\rangle_\nu$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\nu \left\langle |\nabla \boldsymbol{u}|^2 \right\rangle_\nu + \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle \equiv -\varepsilon_\mathrm{D} + \varepsilon_\mathrm{I}$$

Inviscid limit: $\lim_{\nu \to 0} \nu \left\langle |\nabla \boldsymbol{u}|^2 \right\rangle_\nu = \varepsilon_\mathrm{D} > 0$



$\Longrightarrow$ breakdown of Euler time-reversibility

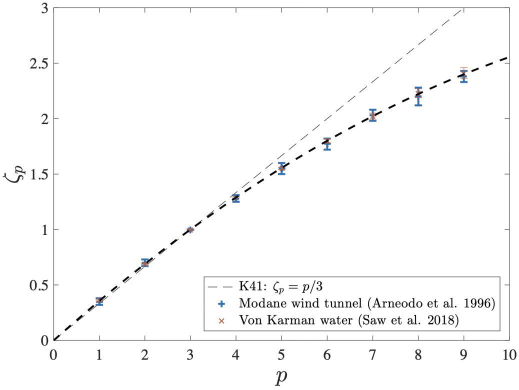
Spatial intermittency

Longitudinal structure functions: $S_p^{\parallel}(r) = \left\langle \left[(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}} \right]^p \right\rangle$

Exact result: 4/5 law
(homogeneity, isotropy, stationarity)

$$S_3^{\parallel}(r) = -\frac{4}{5}\,\varepsilon_I\, r$$

However: $S_p(r) \sim r^{\zeta_p}$ with $\zeta_p \neq p/3$



$\Longrightarrow$ breakdown of simple scale invariance

Lagrangian spontaneous stochasticity

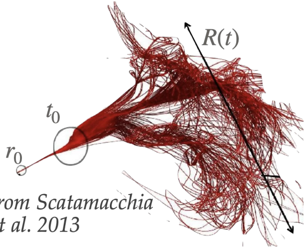
Lagrangian dynamics: $\dfrac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{X}, t), \quad \boldsymbol{X}(0) = \boldsymbol{x}_0$

Velocity field is Hölder, not Lipschitz: $|\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{x}')| \sim |\boldsymbol{x} - \boldsymbol{x}'|^h$ with $h < 1$

Pair separation: $R(t) = |\boldsymbol{X}(t) - \boldsymbol{X}'(t)|, \quad R(0) = r_0$

Short times: $\begin{cases} R(\pm t) \simeq r_0\, \mathrm{e}^{\pm\lambda_\pm t} \\ \text{with } \lambda_-/\lambda_+ \approx 1.3 \text{ and } \lim_{\nu\to 0} \lambda_\pm = \infty \end{cases}$

Long times: $\begin{cases} \left\langle R^2(\pm t) \right\rangle_\nu \simeq g_\pm \varepsilon_\mathrm{I} t^3 \text{ (Richardson dispersion)} \\ \text{with } g_-/g_+ \approx 2 \text{ and } \lim\limits_{\nu\to 0} g_\pm = \text{const} \end{cases}$



from Scatamacchia et al. 2013

Transition probability: $\lim\limits_{|\boldsymbol{x}_0 - \boldsymbol{x}_0'| \to 0} \lim\limits_{\nu \to 0} P_\nu(\boldsymbol{x}, \boldsymbol{x}', t \,|\, \boldsymbol{x}_0, \boldsymbol{x}_0', 0) \neq \delta(\boldsymbol{x} - \boldsymbol{x}')$

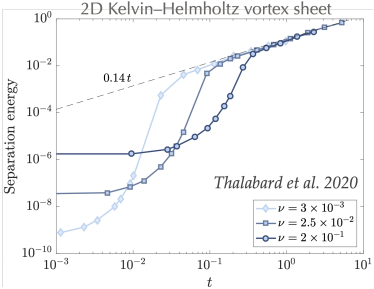
$\Longrightarrow$ stochastic Lagrangian trajectories

Eulerian spontaneous stochasticity

Perturbation of a steady-state solution at $t = 0$: $\boldsymbol{u}'(\boldsymbol{x}, 0) = \boldsymbol{u}(\boldsymbol{x}, 0) + \delta\boldsymbol{u}_0(\boldsymbol{x})$

Growth of the root-mean-squared separation: $\delta u(t) = |\boldsymbol{u}(\cdot, t) - \boldsymbol{u}'(\cdot, t)|_{L_2}$

Short times: $\begin{cases} \delta u(t) \simeq \delta u(0)\, \mathrm{e}^{\lambda^{\mathrm{E}} t} \\ \text{with } \lim_{\nu \to 0} \lambda^{\mathrm{E}} = \infty \end{cases}$



Long times: $\langle \delta u^2(t) \rangle_\nu \simeq C\, \varepsilon_{\mathrm{I}}\, t$

Lorenz's "real" butterfly effect, less predictable than any chaotic system. $\lim_{\nu \to 0} C$? Universality?

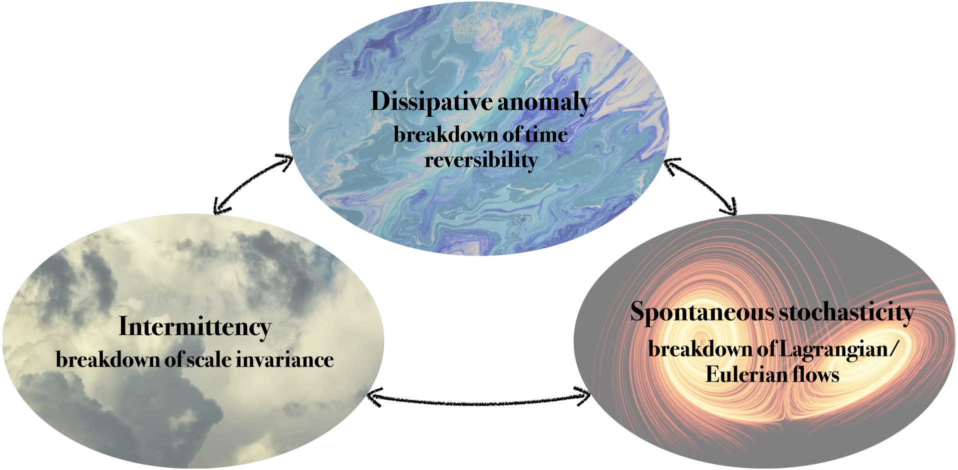
Transition probability: $\lim_{|\boldsymbol{u}(0) - \boldsymbol{u}'(0)| \to 0} \lim_{\nu \to 0} P(\boldsymbol{u}, \boldsymbol{u}', t \,|\, \boldsymbol{u}_0, \boldsymbol{u}'_0, 0) \neq \delta(\boldsymbol{u} - \boldsymbol{u}')$

$\leftrightarrow$ non-unicity of weak Euler solutions and lack of selection principle when $\nu \to 0$.

$\Longrightarrow$ breakdown of standard Euler dynamics

Toward a unified vision?

Subtle interlinks between turbulent solutions and limiting inviscid dynamics



Can simpler models shed some lights?

- Pressureless gas: $\boxed{\partial_t u + u\,\partial_x u = \nu\,\partial_x^2 u + f}$ $\quad x \in \Omega = \mathbb{T}$ or $\mathbb{R}$

  $u(x,t_0) = u_0(x)$ and f random forcing concentrated at scale L_f.

- Hamilton–Jacobi: $\boxed{\partial_t \Psi + \frac{1}{2}\,|\partial_x \Psi|^2 = \nu\,\partial_x^2 \Psi + F}$ with $f = \partial_x F$, $u = \partial_x \Psi$

- Hopf–Cole transformation: $\theta = \mathrm{e}^{-\frac{1}{2\nu}\Psi} \quad \Longrightarrow \quad \boxed{\partial_t \theta = \nu \partial_x^2 \theta - \frac{1}{2\nu} F\,\theta}$

- Feynman-Kac (path integral): $\theta(x,t) = \mathbb{E}_Z\left[\mathrm{e}^{-\frac{1}{2\nu}\Psi_0(Z_{t-t_0}) - \frac{1}{2\nu}\int_{t_0}^{t} F(Z_{t-s},s)\,\mathrm{d}s}\right]$

  with $\quad \mathrm{d}Z_s = \sqrt{2\nu}\,\mathrm{d}W_s, \quad Z_0 = x.$

- Inviscid limit $\nu \to 0$: variational principle $\boxed{\Psi(x,t) = \inf_{\gamma(\cdot):\gamma(t)=s} \left[\Psi_0(\gamma(t_0)) + \mathcal{A}[\gamma; t_0, t]\right]}$

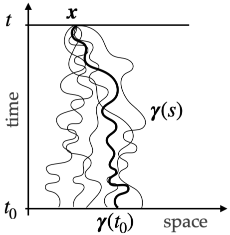
$$\text{with } \mathcal{A}[\gamma; t_0, t] = \int_{t_0}^{t} \left[\tfrac{1}{2}\dot{\gamma}^2(s) + F(\gamma(s), s)\right] \mathrm{d}s$$

$d = 1$: Hopf '50, Lax '57, Oleinik '57, Mather '82, Aubry '83, E *et al.* '00.

$d > 1$: Kruzhkov '75, Lions '82, Iturriaga & Khanin '03

Lagrangian picture

$$u(x,t) = \partial_x \inf_{\gamma(\cdot):\gamma(t)=s} \left[\Psi_0(\gamma(t_0)) + \mathcal{A}[\gamma; t_0, t]\right] \text{ with } \mathcal{A}[\gamma; t_0, t] = \int_{t_0}^{t} \left[\tfrac{1}{2}\dot{\gamma}^2(s) + F(\gamma(s), s)\right] \mathrm{d}s$$



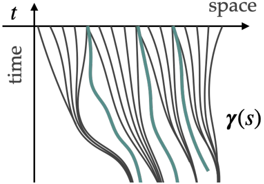
Euler-Lagrange equations:

$$\begin{cases} \dot{\gamma} = v \\ \dot{v} = \partial_x F(\gamma, t) \end{cases}$$

with BC: $\gamma(t) = x$

$v(t_0) = u_0(\gamma(t_0))$.

F smooth (C^3) $\Rightarrow$ infimum attained almost everywhere for a unique $\gamma(\cdot)$



More than one minimizer = shock

The variational principle selects a unique viscosity solution.

Statistically stationary regime

Limit $t_0 \to -\infty$: $\quad u(x,t) = \partial_x \inf_{\gamma(\cdot):\gamma(t)=s} \mathcal{A}_t[\gamma] \quad$ with $\mathcal{A}_t[\gamma] = \int_{-\infty}^{t} \left[\frac{1}{2}\dot{\gamma}^2(s) + F(\gamma(s),s)\right] \mathrm{d}s$

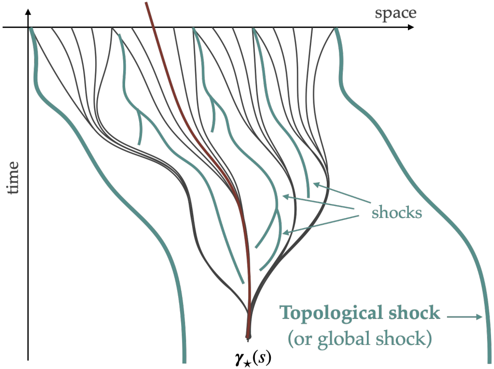
NB: if Ω is compact, the limits $\nu \to 0$ and $t_0 \to \infty$ commute

Key properties:

- minimizers do not intersect at $s < t$
- they are unique, except on a countable set

Ω compact $\Rightarrow$ there exists a unique **global minimizer** $\gamma_\star$ that minimizes $\mathcal{A}_t$ $\forall t$.
$\gamma_\star$ is a **hyperbolic** trajectory of the Lagrangian dynamics: $\forall \gamma$ and $\forall s \ll t$

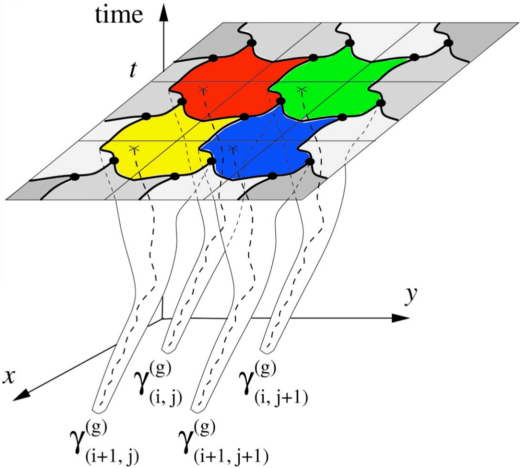
$$|\gamma_\star(s) - \gamma(s)| \leq C|\gamma_\star(t) - \gamma(t)|\, \mathrm{e}^{\lambda(s-t)}$$



By periodicity, the left-most and right-most minimizers approaching $\gamma_\star$ emanate from the same location. They define the **topological shock**.

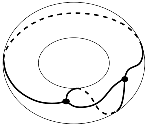
Topological shocks in higher dimensions

Unfold to universal cover

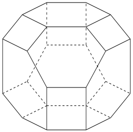


Lattice of global minimizer

For $d = 2$: hexagonal tiling
+ metamorphoses

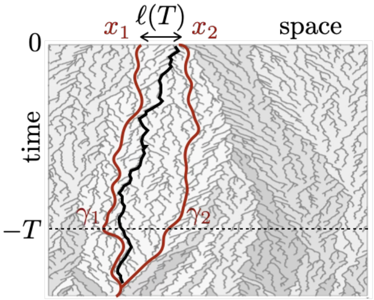


For $d > 2$: less topological constraints
⇒ non-unique tiling



Forcing at scale $L_f \ll L$. Consider intermediate times $L_f/U \ll T \ll L/U$.

T-global shock: shock that is older than T.
Density $\rho(T) \sim T^{-\alpha} \;\Leftrightarrow\; \ell(T) = |x_2 - x_1| \sim T^{\alpha}$



Displacement of minimizers = drift + diffusion
$\ell(T) \sim A\,U\,T + B\,T^{1/2}$ with $U \approx \frac{1}{\ell(T)} \int_{x_1}^{x_2} u(x,t)\,\mathrm{d}x$
$\ell(T) \gg L_f$: sum of independent variables $U \sim \ell^{-1/2}$

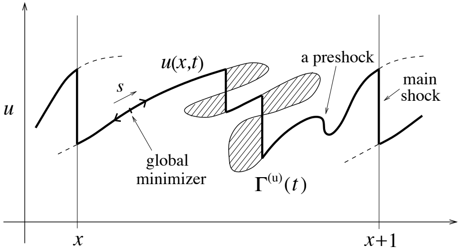
Drift is dominant and $\ell \sim \ell^{-1/2} T$ and $\alpha = 2/3$.

$\ell \sim T^{2/3}$ and $\delta\Psi = \int_{x_1}^{x_2} u(x,t)\,\mathrm{d}x \sim \ell^{1/2} \sim T^{1/3}$ = KPZ scaling
but possibly not same universality class. . .

Assume now that $L_f \sim L$

Convergence of all minimizers to the hyperbolic global minimizer $\gamma_\star$ implies that

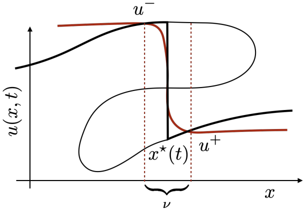
- The statistically steady solution is unique and convergence is exponentially fast
- The graph of the solution $(x, u(x,t))$ belongs to the unstable manifold $\Gamma^{(\mathrm{u})}$ of $\gamma_\star$



- The set of shocks is finite
- The solution is piecewise smooth ($f \in C^r \implies \Gamma^{(\mathrm{u})} \in C^{r-1}$)

(E, Khanin, Mazel, Sinai '00)

- Smoothness properties of the solution allow one to estimate shock contributions:



$$y = \frac{x - x^\star(t)}{\nu} \qquad s(t) = u^- - u^+$$

Inner expansion: $u(x,t) = v_0(y,t) + \nu v_1(y,t) + \nu^2 v_2(y,t) + \cdots$

$$\Longrightarrow \quad v_0(y,t) = \frac{1}{2}\left(u^+ + u^-\right) - \frac{s(t)}{2}\tanh\frac{s(t)\,y}{4}$$

- Shocks dissipate kinetic energy if $s = u^- - u^+ > 0$

$$\varepsilon_\mathrm{D} = \nu \int (\partial_x u)^2\,\mathrm{d}x \simeq \sum_i \frac{\nu s_i^2}{4} \int \left(\partial_x \tanh\frac{s_i\,(x - x_i^\star)}{4\nu}\right)^2 \mathrm{d}x = \frac{\rho_\mathrm{s} L}{12}\left\langle s^3 \right\rangle$$

- Approach extended to other quantities: e.g., PDF $p(\xi)$ of velocity gradients $\partial_x u$

$-\partial_\xi(\xi^2 p) - \xi\, p = \mathcal{D}_\nu(\xi) + \chi_1\, \partial_\xi^2 p$ (E & Vanden-Eijnden '00)

with $\mathcal{D}_\nu = \nu\partial_\xi\left(\langle \partial_x^3 u | \partial_x u = \xi\rangle\, p\right) \xrightarrow{\nu\to 0} \frac{\rho_\mathrm{s}}{2}\int_0^\infty s[p^+(\xi,s) + p^-(\xi,s)]\,\mathrm{d}s$

Spatial intermittency

- Equivalent of the $4/5$ law: $S_3(\ell) = \left\langle \delta u^3 \right\rangle \simeq -12\,\varepsilon_{\mathrm{I}}\,\ell, \quad \delta u = u(x+\ell) - u(x)$
- Piecewise smoothness and finite number of shocks

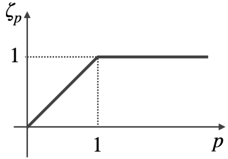
$$\Longrightarrow \; p(\delta u;\ell) \simeq \underbrace{\frac{1}{\ell}\,p_{\partial_x u}\left(\frac{\delta u}{\ell}\right)}_{\text{from smooth pieces}} + \underbrace{\rho_{\mathrm{s}}\ell\,p_{\mathrm{size}}(-\delta u)}_{\text{from shocks}}$$

- Structure functions

$$S_p(\ell) = \langle \delta u^p \rangle \simeq \ell^p \int \xi^p\, p_{\partial_x u}(\xi)\,\mathrm{d}\xi + (-1)^p \rho_{\mathrm{s}}\ell \int_0^\infty s^p p_{\mathrm{size}}(s)\,\mathrm{d}s$$

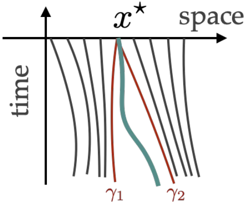
Bifractal scaling:

$$\boxed{S_p(\ell) \sim \ell^{\zeta_p} \text{ with } \zeta_p = \begin{cases} p & p \leq 1 \\ 1 & p > 1 \end{cases}}$$



- Spatial intermittency is **geometric** in origin: isolated shocks (codimension one) dominate high-order statistics, leading to saturation of scaling exponents

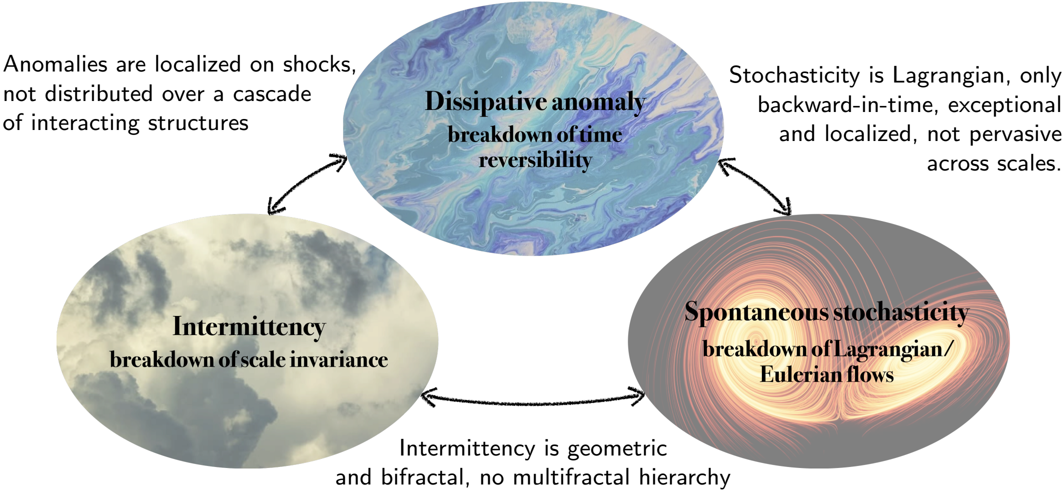
- **Lagrangian trajectories:** non-uniqueness of backward characteristics at shocks
  When $\nu \to 0$ (Eyink & Drivas '15):
  $P_0(x, -t|x^*, 0) = \frac{1}{2}\delta(x - \gamma_1(t)) + \frac{1}{2}\delta(x - \gamma_2(t))$
  Direct relation between stochasticity of Lagrangian trajectories and anomalous dissipation, and extreme time asymmetry.
  Occurs however on a set of zero space-time measure.



- **Eulerian fields:** The construction of the statistically stationary state implies a one-force/one-solution principle. Viscosity solutions are unique and the dynamics is not even chaotic.
  No Eulerian spontaneous stochasticity at the level of the velocity field u, despite stochasticity in the underlying Lagrangian dynamics.

Lessons from Burgers turbulence

Dissipation, intermittency, and stochasticity emerge from singular dynamics (shocks).



Open questions in Burgers turbulence

- **Universality with respect to small-scale regularization**
  Does the inviscid-limit, shock-dominated Burgers phenomenology persist under alternative regularizations (e.g. hyperviscosity, stochastic noise)? What happens if the small-scale dynamics is fundamentally altered, potentially producing spatiotemporal chaos as in Kuramoto–Sivashinsky?

- **Scale-invariant forcing**
  When forcing is scale-invariant, the solution becomes Hölder-regular and shocks can be dense. How to construct the steady-sate? Can this lead to Eulerian spontaneous stochasticity and new intermittency mechanisms?