Entropic repulsion in low temperature lattice models

Joint works with Eyal Lubetzky (NYU) and Reza Gheissari (Northwestern)

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Inhomogeneous Random Systems - January 2026

Sorbonne Université

Definition: Ising/Potts model

- Potts model (β, q): random q-coloring σ of the vertices of $G = (V, E)$ with law

$$\mu(\sigma) = \frac{1}{Z} \exp(-\beta \sum_{x \sim y} \mathbf{1}_{\sigma(\mathbf{x}) \neq \sigma(\mathbf{y})})$$

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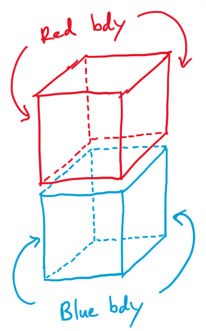
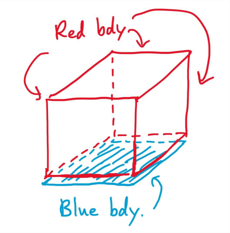
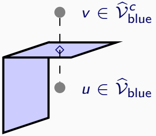
- In this talk:
    1. G = subgraph of $\mathbb{Z}^3$
    2. blue-red (Dobrushin) boundary conditions
    3. low temperature (β sufficiently large)

Two graphs: Cylinder (no floor) and Box (with floor)







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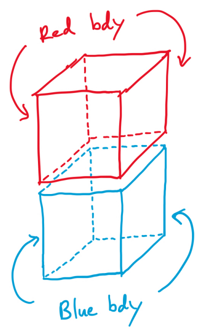
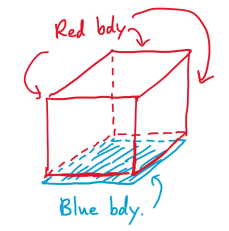
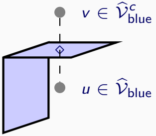
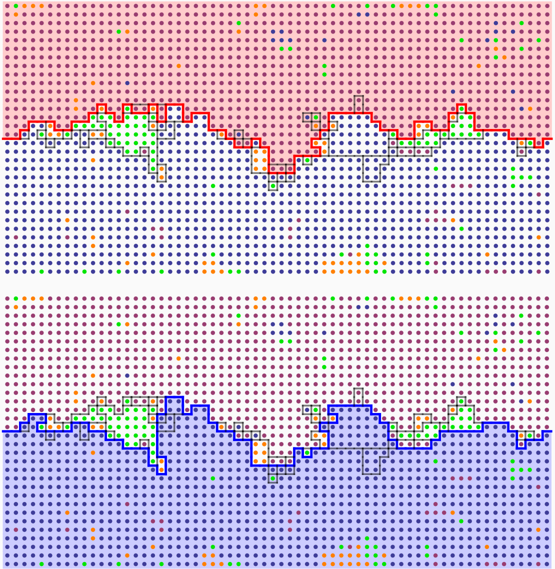




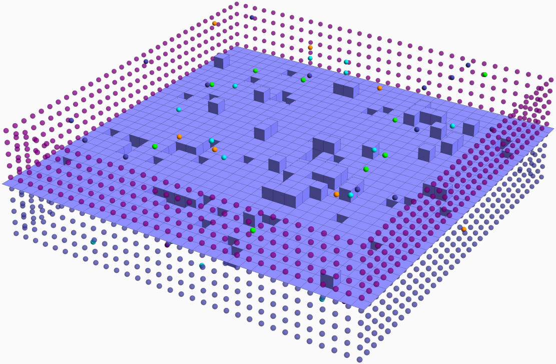


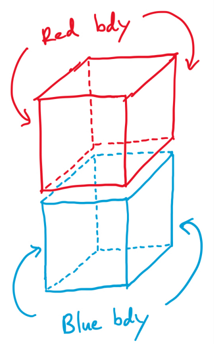
- Interfaces: blue vs. non-blue ($\mathcal{I}_{\text{blue}}$) and red vs. non-red ($\mathcal{I}_{\text{red}}$)
- Def: Take connected component of blue vertices containing the blue boundary, $\mathcal{V}_{\text{blue}}$. Fill in all "holes", $\widehat{\mathcal{V}}_{\text{blue}}$. Interface is set of "dual faces" separating $\widehat{\mathcal{V}}_{\text{blue}}$ from $\widehat{\mathcal{V}}_{\text{blue}}^{c}$.

(Misleading) pictures in 2D cylinder



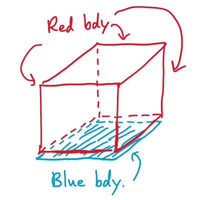
- $G = \Lambda_{n,\text{cyl}} = n \times n \times 2n$
- Model: Layer between oil and water in a cup
- Interface is rigid with maximum of $O(\log n)$
  (Dobrushin '72,'73, Grimmett, Gielis '02)

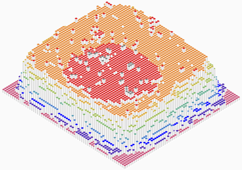


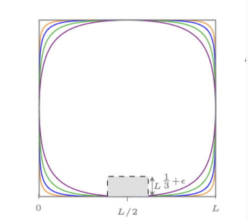


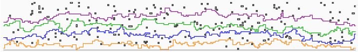
Background: Box (with floor)

- Now add 'hard floor': $G = \Lambda_{n,\text{box}} = n \times n \times n$
- Model: Bacteria/crystals growing on a dish
- Interesting (mostly conjectured) phenomenon:
  1. Entropic repulsion: rigid surface delocalizes
  2. Lattice effect: See Wulff shape at surface boundary
  3. Surface boundary = area tilted random walk ensemble





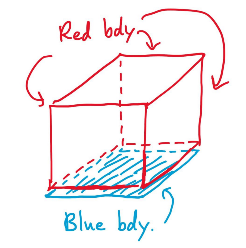


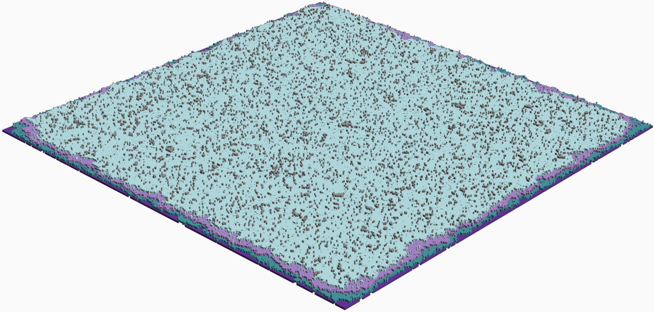


Background: Motivating Question

Question

In $\Lambda_{n,\text{box}}$, what is the average height of $\mathcal{I}_{\text{blue}}$ above the origin?





Background: Entropic repulsion in 3D Potts

In the 3D Ising/Potts model:

- For $q = 2$ (Ising), have delocalization (typical height diverges as $n \to \infty$) (Fröhlich, Pfister '87), no quantitative results
- For $q > 2$: no rigorous results

New results (Logarithmic delocalization)

Theorem (C., Gheissari, Lubetzky '25)

Fix $q \geq 2$ and β large enough. Consider the q-state Potts model on the $n \times n \times n$ box $\Lambda_{n,\mathrm{box}}$ with boundary conditions blue on its bottom side and red elsewhere. There exists a constant γ and a sequence $\epsilon_\beta \downarrow 0$ as $\beta \uparrow \infty$ such that w.h.p,

$$|\{f \in \mathcal{I}_{\mathrm{blue}} : \mathrm{hgt}(f) \notin [\tfrac{1-\epsilon_\beta}{\gamma} \log n, \tfrac{1}{\gamma} \log n]\}| \leq \epsilon_\beta n^2 .$$

- More on γ later

Background: Entropic repulsion in height functions

Entropic repulsion in (2+1)D height function models:

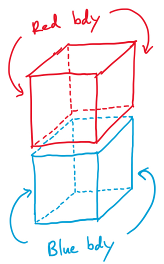
	Height in box (condition ≥ 0)	max in cyl (no conditioning)
SOS	$\frac{1}{4\beta} \log n$	$\frac{1}{2\beta} \log n$
$\mathbb{Z}$-DG	$\frac{1}{2\sqrt{\pi\beta}} \sqrt{\log n \log \log n}$	$\frac{1}{\sqrt{2\pi\beta}} \sqrt{\log n \log \log n}$
$\mathbb{R}$-DG	$\frac{2\sqrt{2}}{\sqrt{\pi}} \log n$	$\frac{2\sqrt{2}}{\sqrt{\pi}} \log n$

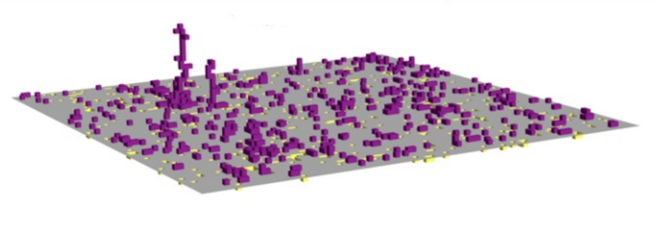
- SOS - 1:2 scaling (Bricmont, El-Mellouki, Fröhlich '86, Caputo et al. '14, '16)
- $\mathbb{Z}$-DG - 1:$\sqrt{2}$ scaling (BEF '86, Lubetzky, Martinelli, Sly '16)
- $\mathbb{R}$-DG - 1:1 scaling (Bolthausen, Deuschel, Giacomin '01, B., D., Zeitouni '11)

Switch to no-floor setting

Question

In $\Lambda_{n,\text{cyl}}$, what is the probability that $\text{ht}_x(\mathcal{I}_{\text{blue}}) \leq -h$? What is the min height of $\mathcal{I}_{\text{blue}}$?





New results (Potts extrema)

Theorem

($q = 2$ Gheissari, Lubetzky '22), ($q > 2$ C., Lubetzky '24)

Fix $q \geq 2$ and β large enough. For the Potts model on $\Lambda_{n,\text{cyl}}$ with Dobrushin boundary conditions, let $\text{Min}_n^{\text{blue}}, \text{Max}_n^{\text{blue}}$ be the min, max heights of $\mathcal{I}_{\text{blue}}$. Both $\text{Min}_n^{\text{blue}}, \text{Max}_n^{\text{blue}}$ are tight around their means, and $\exists \gamma, \gamma'$ such that

$$\mathbb{E}[\text{Min}_n^{\text{blue}}] = \tfrac{2}{\gamma} \log n + O(1)$$

$$\mathbb{E}[\text{Max}_n^{\text{blue}}] = \tfrac{2}{\gamma'} \log n + O(1)$$

Moreover, for any $h \ll n$,

$$\gamma = -\lim_{n \to \infty} \frac{1}{h_n} \log \mu_{n,\text{cyl}}(\text{ht}_o(\mathcal{I}_{\text{blue}}) \leq -h_n)$$

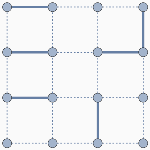
$$\gamma' = -\lim_{n \to \infty} \frac{1}{h_n} \log \mu_{n,\text{cyl}}(\text{ht}_o(\mathcal{I}_{\text{blue}}) \geq h_n)$$

Definition: Random Cluster/FK model

- Random Cluster (FK) model (p, q): random edge configuration ω on edges of $G = (V, E)$ with law

$$\pi(\omega) = \frac{1}{Z} p^{|\omega|}(1-p)^{|E \setminus \omega|} q^{\kappa(\omega)}$$

where $\kappa(\omega)$ = number of connected components in ω, called *clusters*.



- Edges of G are *open/wired* if they are present in ω, and *closed* otherwise.

Edwards-Sokal Coupling

- Joint measure

$$\mathbf{P}(\sigma, \omega) = \frac{1}{Z} p^{|\omega|} (1-p)^{|E \setminus \omega|} \prod_{[x,y] \in \omega} \mathbf{1}_{\sigma(x)=\sigma(y)}$$

- Edwards-Sokal coupling says for $p = 1 - e^{-\beta}$, this is a coupling of Potts and random cluster (Edwards, Sokal '88)

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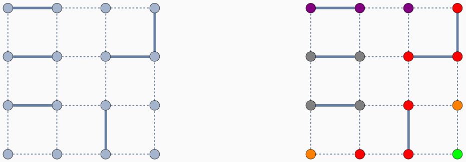
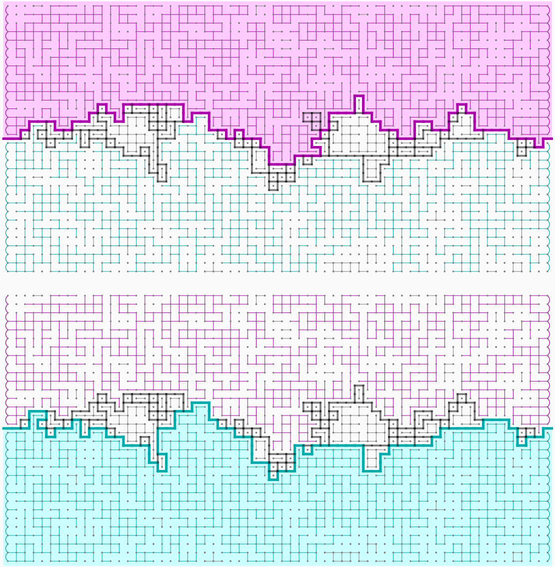
- To get Potts, first sample a random cluster model and then color clusters independently, uniformly at random.



Definition: Random-cluster interfaces

- Dobrushin boundary conditions: open in $\mathbb{H}_-$, open in $\mathbb{H}_+$, closed in between. Additionally condition on disconnection between top and bottom boundaries: $\bar{\pi}(\cdot) = \pi(\cdot \mid \mathfrak{D})$
- Two more interfaces: top interface and bot interface
- Def: Take ω-connected component containing the top boundary, $\mathcal{V}_{\mathsf{top}}$. Fill in all "holes", $\widehat{\mathcal{V}}_{\mathsf{top}}$. Interface is set of "dual faces" separating $\widehat{\mathcal{V}}_{\mathsf{top}}$ from $\widehat{\mathcal{V}}_{\mathsf{top}}^c$.

Misleading 2D pictures



New results (Random cluster extrema)

Theorem 2 (C., Lubetzky '24)

Consider the random cluster model on $\Lambda_{n,\mathrm{cyl}}$ with Dobrushin boundary conditions, $q \geq 1$, and $p = 1 - e^{-\beta}$ for $\beta > \beta_0$. Let $\mathrm{Min}_n^{\mathrm{bot}}, \mathrm{Max}_n^{\mathrm{bot}}$ be the min, max heights of $\mathcal{I}_{\mathrm{bot}}$. Both $\mathrm{Min}_n^{\mathrm{bot}}, \mathrm{Max}_n^{\mathrm{bot}}$ are tight around their means, and $\exists \alpha, \alpha'$ s.t.

$$\mathbb{E}[\mathrm{Min}_n^{\mathrm{bot}}] = (\tfrac{2}{\alpha} + o(1)) \log n$$
$$\mathbb{E}[\mathrm{Max}_n^{\mathrm{bot}}] = (\tfrac{2}{\alpha'} + o(1)) \log n$$

Moreover, for any $h \ll n$,

$$\alpha = -\lim_{n\to\infty} \frac{1}{h_n} \log \bar{\pi}_{n,\mathrm{cyl}}(\mathrm{ht}_o(\mathcal{I}_{\mathrm{bot}}) \leq -h_n)$$
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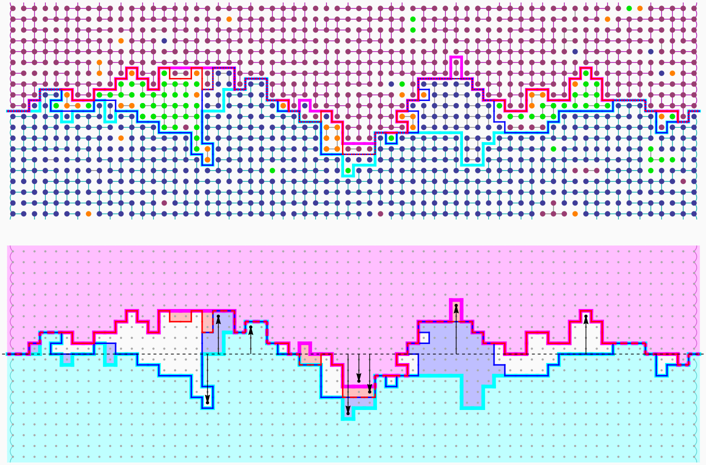
4 maxima, 4 minima, 4 LD rates



Comparison of 4 rates

Theorem (C., Lubetzky '24)

The constants $\alpha, \alpha', \gamma, \gamma'$ governing the asymptotic means of the maxima and minima of 3D Potts and 3D FK interfaces satisfy

$$
\begin{aligned}
4\beta - C \le \alpha \le 4\beta\,, \\
\gamma - \alpha = (1 \pm \epsilon_\beta) e^{-\beta}\,, \\
\gamma' - \alpha = (1 \pm \epsilon_\beta)(q-1) e^{-\beta}\,, \\
\alpha' - \alpha = (1 \pm \epsilon_\beta) q e^{-\beta}\,.
\end{aligned}
$$

In particular, the four rates are distinct.

Switch back to floor setting: Proof ideas for logarithmic delocalization

Question (again)

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Switch back to floor setting: Proof ideas for logarithmic delocalization

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- Prediction from SOS model: height is $O(\log n)$
- Cost of raising entire interface from height $h-1$ to h should be $\approx \exp(-4\beta n)$, gain in entropy should be $\approx (1+e^{-4\beta h})^{n^2} \approx \exp(e^{-4\beta h} n^2)$

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- In actual Potts/Ising model, interaction with the floor is same order as gain in entropy.

Proof ideas: Lower bound height (with floor)

- Claim: easier to lift $\mathcal{I}_{\text{top}}$ in $\Lambda_{n,\text{box}}$ vs. $\Lambda_{n,\text{cyl}}$

$$\frac{\bar{\pi}_{n,\text{box}}(\mathcal{I}_{\text{top}} = \Theta I)}{\bar{\pi}_{n,\text{box}}(\mathcal{I}_{\text{top}} = I)} \Big/ \frac{\bar{\pi}_{n,\text{cyl}}(\mathcal{I}_{\text{top}} = \Theta I)}{\bar{\pi}_{n,\text{cyl}}(\mathcal{I}_{\text{top}} = I)} \overset{?}{\geq} 1$$

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$$\frac{\bar{\pi}_{n,\text{box}}(\mathcal{I}_{\text{top}} = \Theta l)}{\bar{\pi}_{n,\text{box}}(\mathcal{I}_{\text{top}} = l)} \Bigg/ \frac{\bar{\pi}_{n,\text{cyl}}(\mathcal{I}_{\text{top}} = \Theta l)}{\bar{\pi}_{n,\text{cyl}}(\mathcal{I}_{\text{top}} = l)} \overset{?}{\geq} 1$$

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- View box as taking cyl and adding edges with strength $s = 1$ between vertices at height 0. Then interpolate $s \in [0, 1]$.

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- View box as taking cyl and adding edges with strength $s = 1$ between vertices at height 0. Then interpolate $s \in [0, 1]$.
- $\pi^{s}_{n,\text{cyl},\Theta I_\downarrow}(e \text{ is open}) \overset{?}{\geq} \pi^{s}_{n,\text{cyl},I_\downarrow}(e \text{ is open})$

Proof ideas: Lower bound height (with floor)

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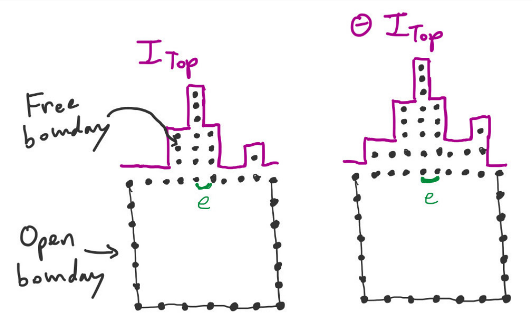

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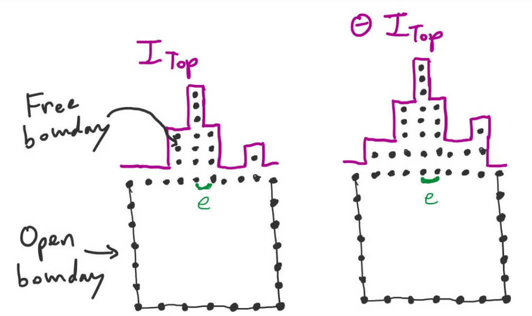

- Inequality is true by monotonicity **if** ΘI lies entirely above I. **Not true in general** because of overhangs.

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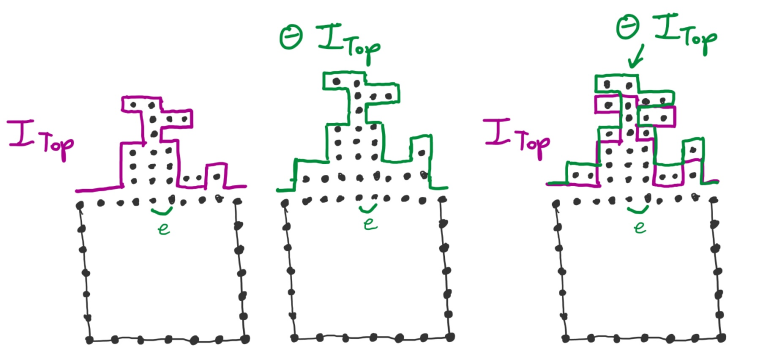

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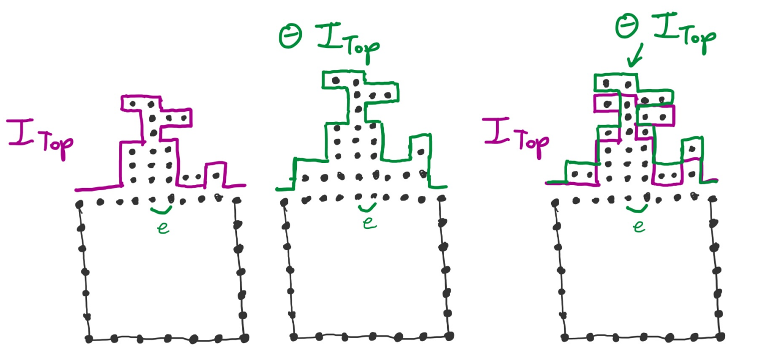


- Remedy: Instead of lifting by 1, lift by $\log n$. It is rarer to be $\log\log n$ below the correct height than to be $\log n$ above it!

Proof ideas: Lower bound height (with floor)

- Conclusion: Can use “lifting argument” to prove typical height of $\mathcal{I}_{\mathsf{top}}$ is $\geq \frac{1-\epsilon_\beta}{\gamma} \log n$
- Same must be true for $\mathcal{I}_{\mathsf{blue}}$ by rigidity - at most places, $\mathcal{I}_{\mathsf{blue}}$ and $\mathcal{I}_{\mathsf{top}}$ coincide

A paradoxical result?

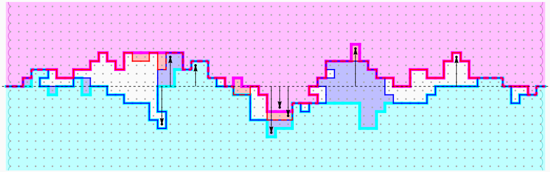
- Consider the joint FK-Potts model on $\Lambda_{n,\text{box}}$
- Conjectured 1:2 scaling predicts:
  - $\mathcal{I}_{\text{blue}}$ at height $\frac{1}{\gamma}\log n$
  - $\mathcal{I}_{\text{red}}$ at $\frac{1}{\gamma'}\log n$
  - $\mathcal{I}_{\text{bot}}$ at $\frac{1}{\alpha}\log n$
  - $\mathcal{I}_{\text{top}}$ at $\frac{1}{\alpha'}\log n$

  Rigidity at large β says there is just one height. But we proved the four rates are different!

A paradoxical result?

One of the following must be true, and they would all be very interesting!

1. The conjectured 1:2 scaling from SOS actually fails for 3D Ising/Potts
2. The conjectured 1:2 scaling from SOS still holds for 3D Ising. This means the FK-Potts model naturally "picks" γ as the governing rate over γ', α, α'



Thank you!