Depinning, sandpiles and hyperuniformity

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IHP, January 2026

PRL 133 (2024) 0671032, arXiv:2401.09123

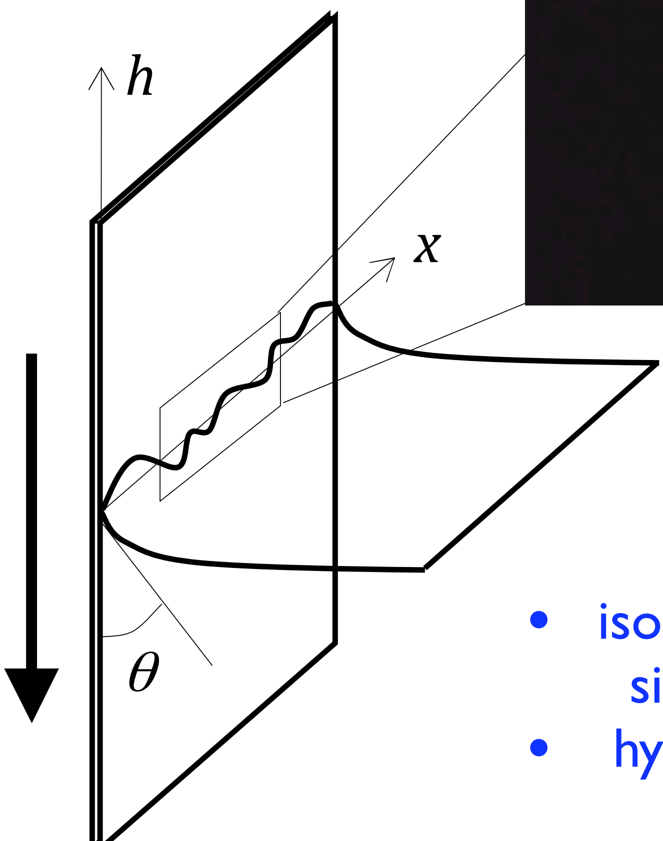
http://www.phys.ens.fr/~wiese/

Review: arXiv:2102.01215

Rep. Prog. Phys. 85 (2022) 086502

Depinning example: Contact line wetting

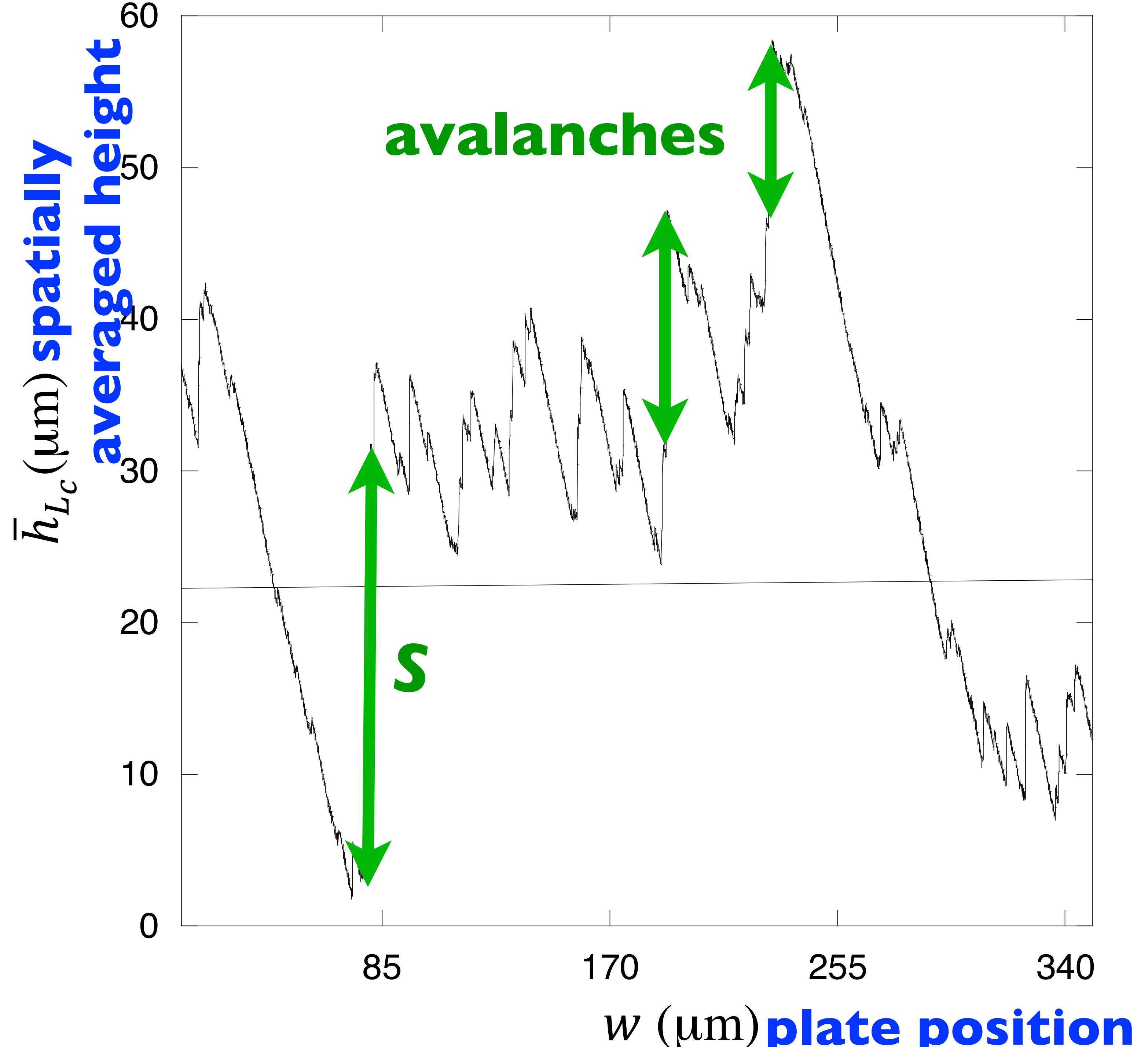
(C) E. Rolley



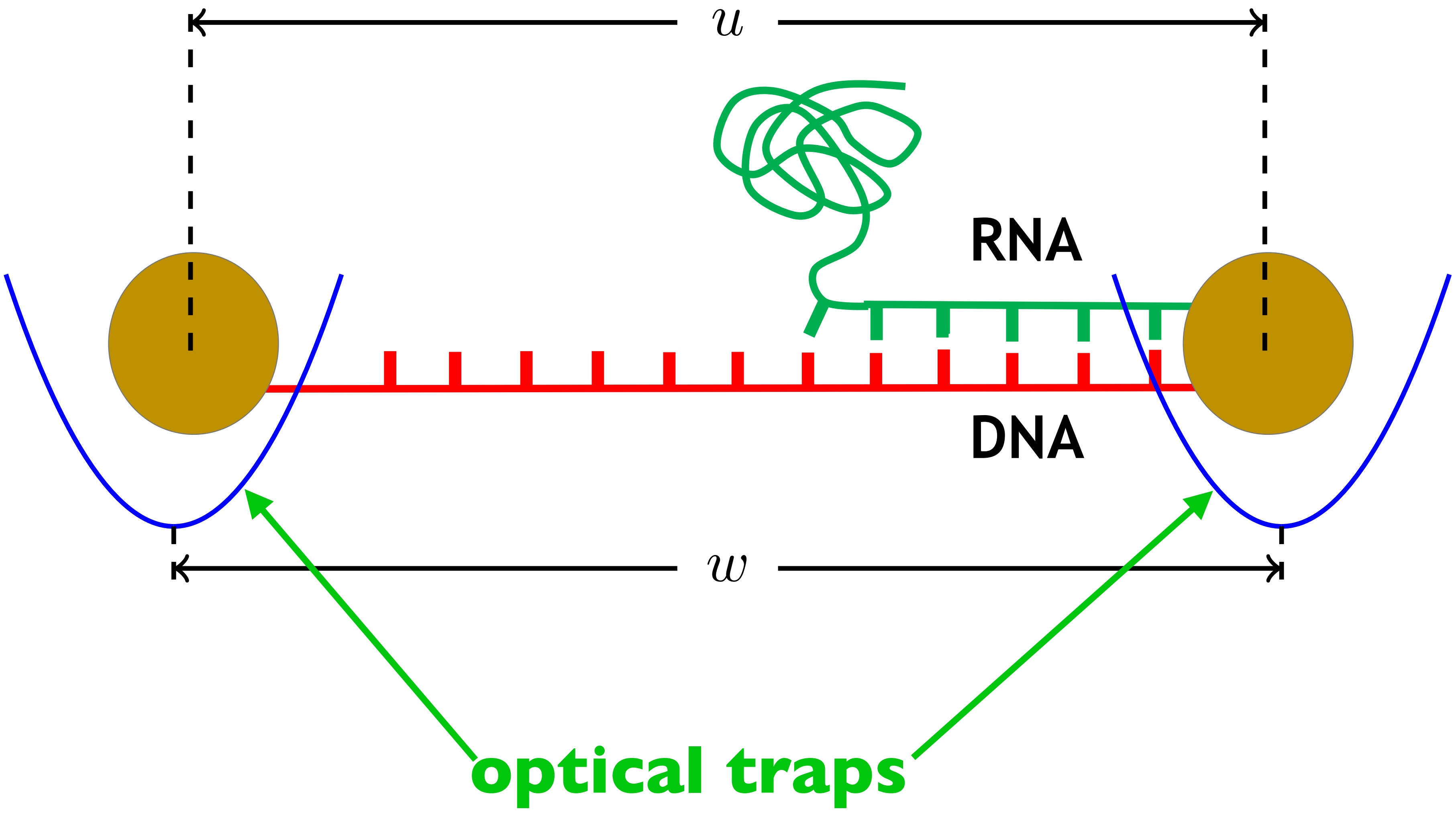
avalanche motion

- isobutanol on a randomly silanized silicon wafer
- hydrogen on disordered Cesium substrate

height jumps = avalanches

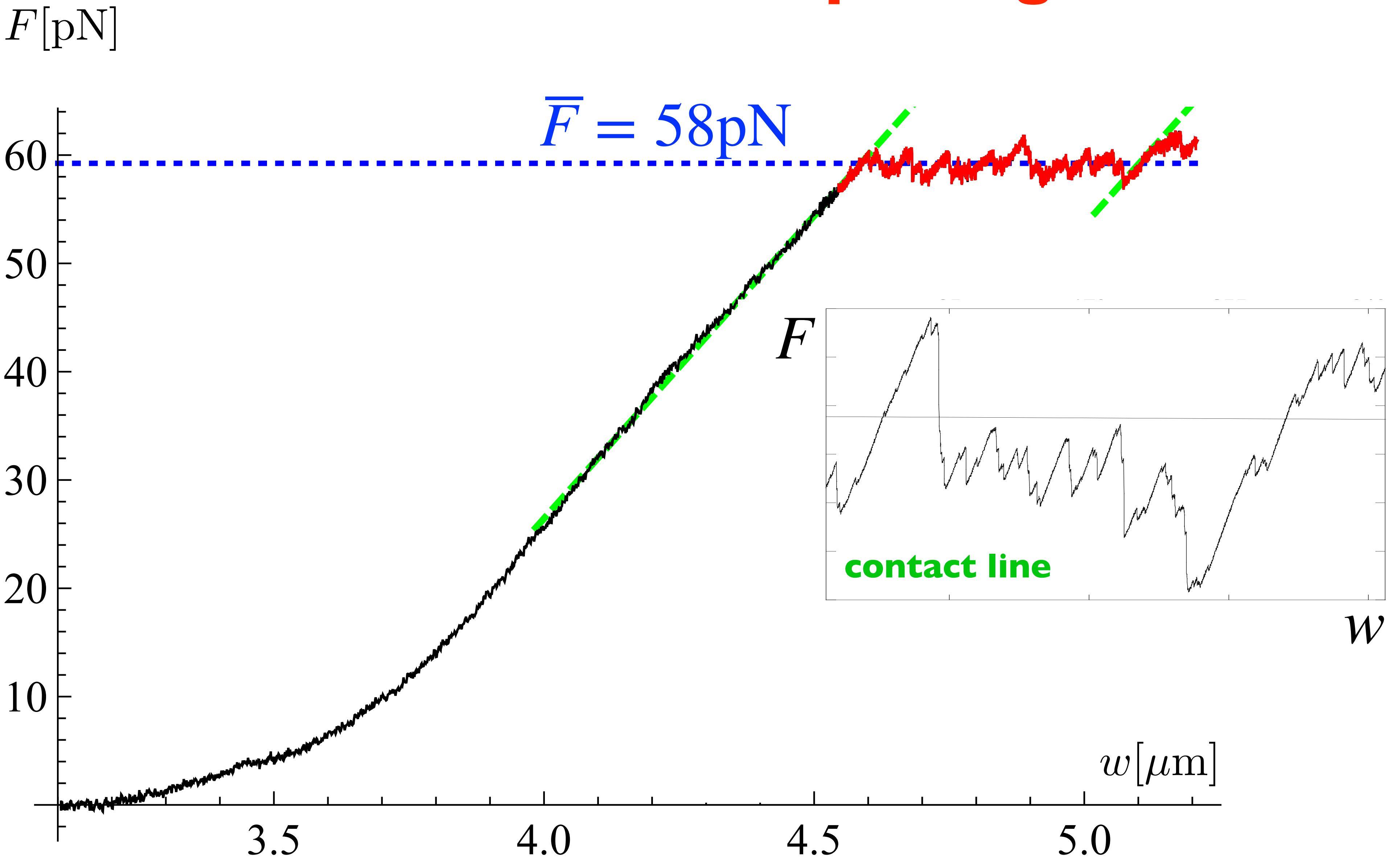


Peeling of an RNA/DNA double helix



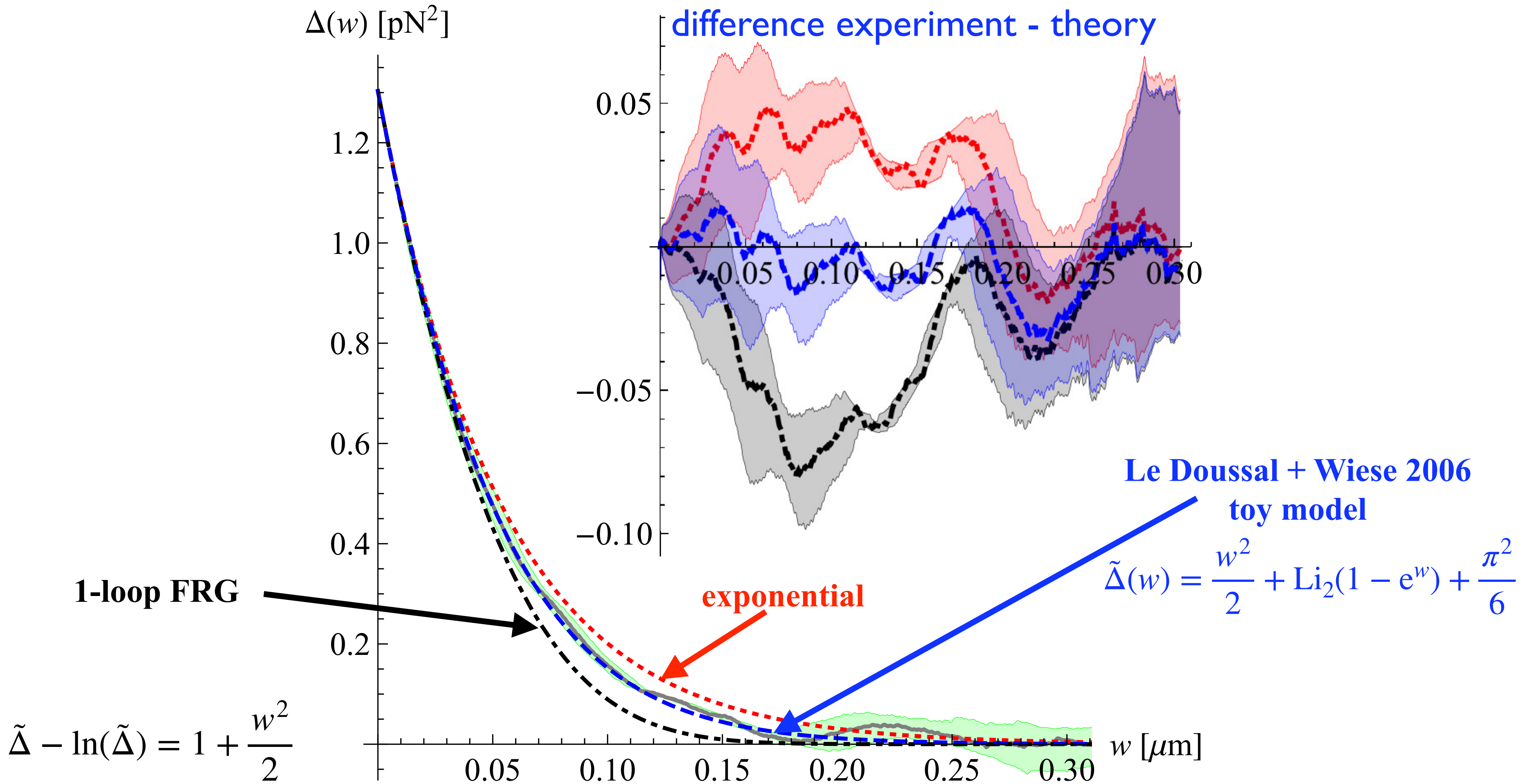
At rest: $F_w = m^2(u_w - w)$

Force as a function of distance for RNA/DNA peeling



Force-force correlations

$$\Delta(w-w') := \overline{F_w F_{w'}}^{c} \equiv \overline{F_w F_{w'}} - \overline{F_w}\ \overline{F_{w'}}$$



Field theory background

Equation of motion (for SR elasticity for simplicity)

height of the interface

$w = vt$

$$\partial_t u(x,t) = \nabla^2 u(x,t) + m^2[w - u(x,t)] + F(x, u(x,t))$$

Forces are drawn from a **Gaussian**, and have correlations

$$\overline{F(x,u)F(x',u')}^{\,\mathrm{c}} = \delta^d(x-x')\Delta(u-u')$$

Field theory (MSR)

$$\mathcal{S}[\tilde{u},u] = \int_{x,t} \tilde{u}(x,t)\Big[\partial_t u(x,t) - \nabla^2 u(x,t) + m^2\Big(u(x,t) - w\Big)\Big]$$

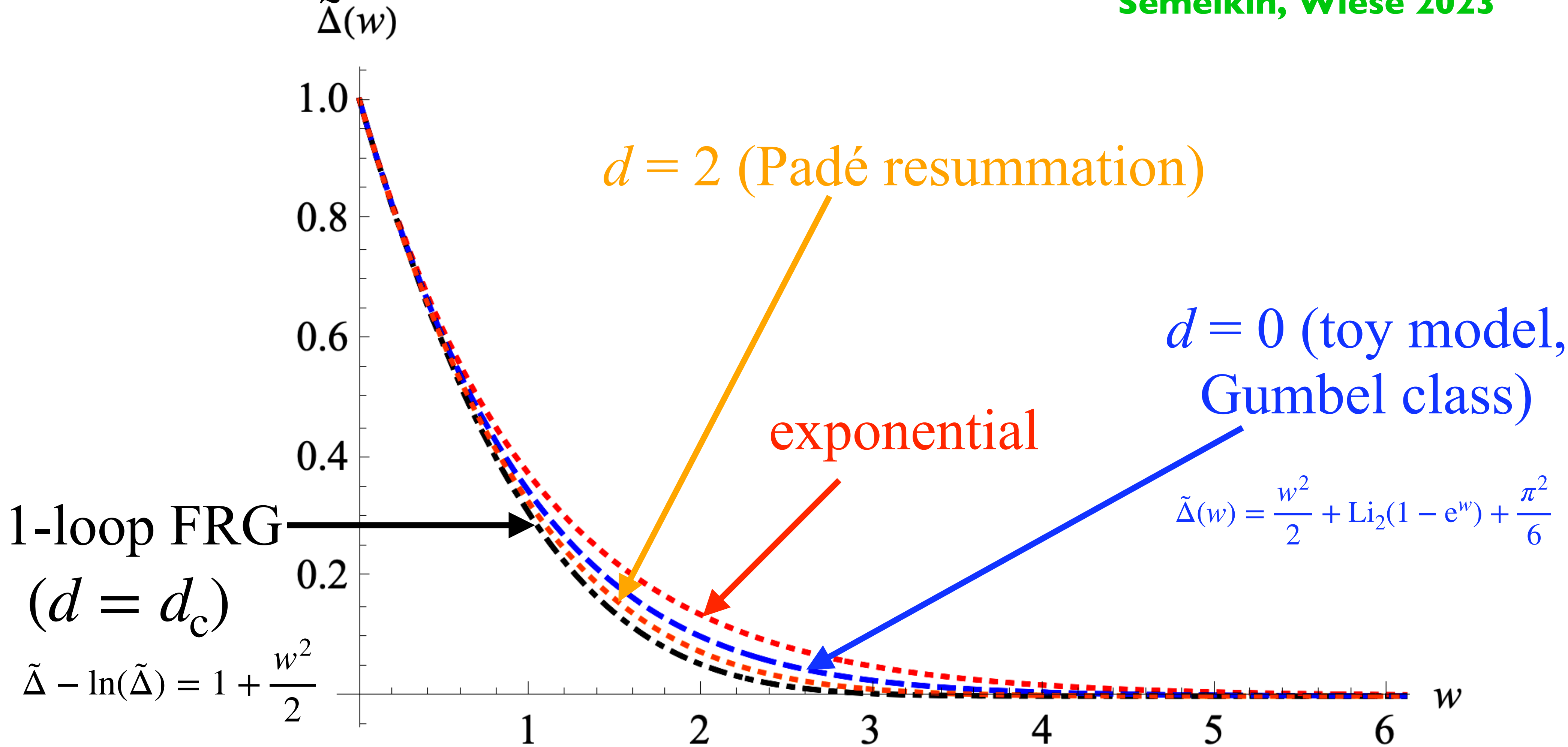
$$-\frac{1}{2}\int_{x,t,t'} \tilde{u}(x,t)\tilde{u}(x,t')\Delta\Big(u(x,t) - u(x,t')\Big)$$

renormalise

Renormalization of (rescaled) disorder

$$-\frac{m\mathrm{d}}{\mathrm{d}m}\tilde{\Delta}(w) = (\epsilon - 2\zeta)\tilde{\Delta}(w) + \zeta w \tilde{\Delta}'(w) - \frac{1}{2}\partial_w^2 \left[\tilde{\Delta}(w) - \tilde{\Delta}(0)\right]^2$$
$$+\frac{1}{2}\partial_w^2 \left\{ \left[\tilde{\Delta}(w) - \tilde{\Delta}(0)\right] \tilde{\Delta}'(w)^2 + \tilde{\Delta}'(0^+)^2 \tilde{\Delta}(u) \right\}$$

Chauve, Le Doussal, Wiese 2004
Semeikin, Wiese 2023



Renormalization of disorder at 3 loops

Semeikin, Wiese 2023

$$\partial_\ell \tilde\Delta(w) = (\epsilon - 2\zeta)\tilde\Delta(w) + \zeta w \tilde\Delta'(w) - \partial_w^2\left[\frac{1}{2}(\tilde\Delta(0) - \tilde\Delta(w))^2\right]$$
$$-\partial_w^2\left[\left(-\frac{1}{2} - \frac{\epsilon}{4} + C_3\epsilon\right)\left(\tilde\Delta'(0^+)^2\tilde\Delta(w) + (\tilde\Delta(w) - \tilde\Delta(0))\tilde\Delta'(w)^2\right)\right]$$
$$-\partial_w^2\left[\frac{3}{4}\zeta(3)\left(\tilde\Delta'(w)^4 - 2\tilde\Delta'(0^+)^2\tilde\Delta'(w)^2 + 8\tilde\Delta'(0^+)^2\tilde\Delta''(0)\tilde\Delta(w)\right)\right.$$
$$+ 2\tilde\Delta'(w)^2\left(\tilde\Delta'(0^+)^2 + (\tilde\Delta(w) - \tilde\Delta(0))\tilde\Delta''(w)\right)$$
$$\left.+ C_3\left((\tilde\Delta(w) - \tilde\Delta(0))^2\tilde\Delta''(w)^2 - \frac{1}{2}\tilde\Delta'(w)^4 + (\tilde\Delta(0) - \tilde\Delta(w))\tilde\Delta'(w)^2\tilde\Delta''(w) - 6\tilde\Delta'(0^+)^2\tilde\Delta''(0)\tilde\Delta(w)\right)\right]$$
$$+2\tilde\Delta'(0^+)^2\tilde\Delta''(w)^2 + \mathcal{O}(\tilde\Delta^5) \tag{40}$$

$$C_3 = \frac{\psi'(\frac{1}{3})}{6} - \frac{\pi^2}{9}$$

$$\zeta = \frac{\epsilon}{3} + 0.047\,770\,971\,5\epsilon^2 - 0.068\,354\,4(2)\epsilon^3 + O(\epsilon^4).$$

Critical force at 3-loop order is universal !

$$f_c = f_0 - \mathscr{B}\rho_m m^2 + \mathcal{O}(m^2),$$

$$\mathscr{B} = 1 + 0.070061\epsilon + 0.0127138\epsilon^2 + \mathcal{O}(\epsilon^3)$$

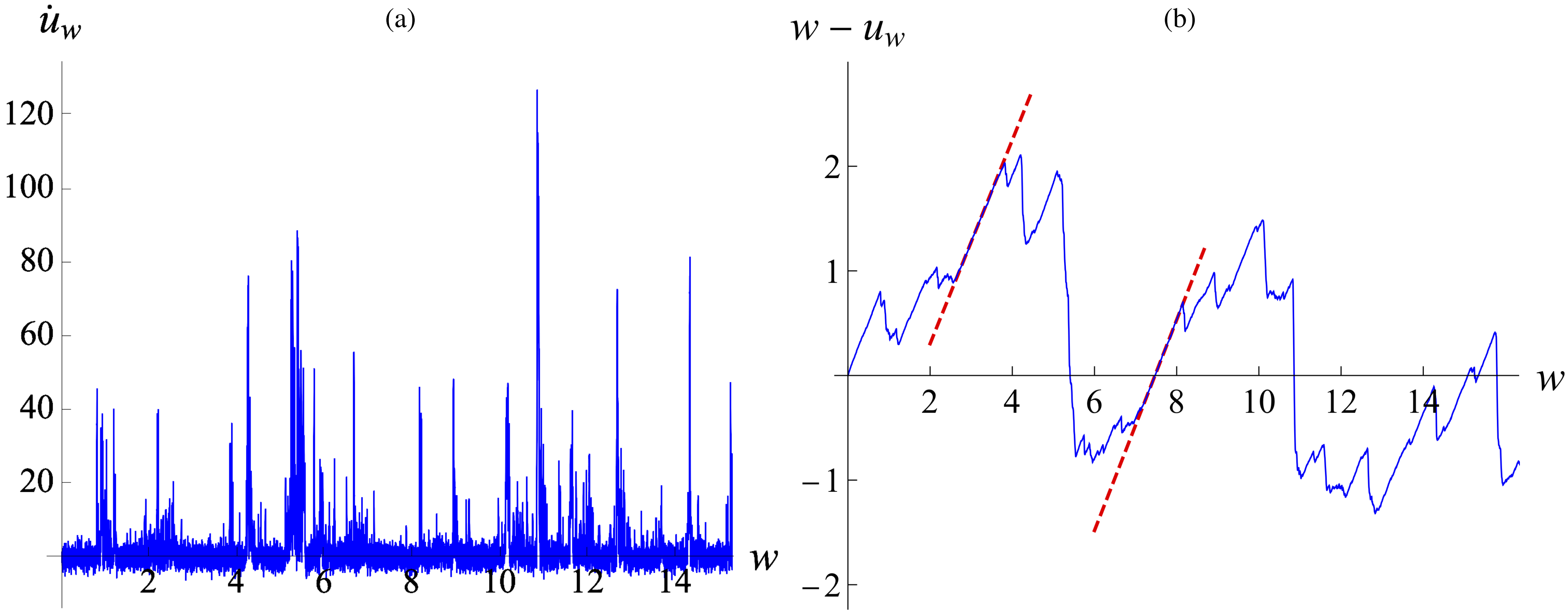
	$d=1$	$d=2$	$d=3$	$d=4$
$\mathcal{B}$ (direct)	1.32	1.19	1.08	1
$\zeta\mathcal{B}$ (Padé-Borel)	1.21	0.78	0.374	0
$\mathcal{B}$ (using $\zeta\mathcal{B}$)	0.96	1.036	1.048	1
$\mathcal{B}$ (estimate and error bars)	1.3(4)	1.1(1)	1.06(2)	1
$\mathcal{B}$ (numerics)	1.8(2)	-	-	-

Who can measure
it for me ?

Magnetic domain walls $(d = 2)$

(data by F. Bohn, G. Durin, R.L. Sommer)

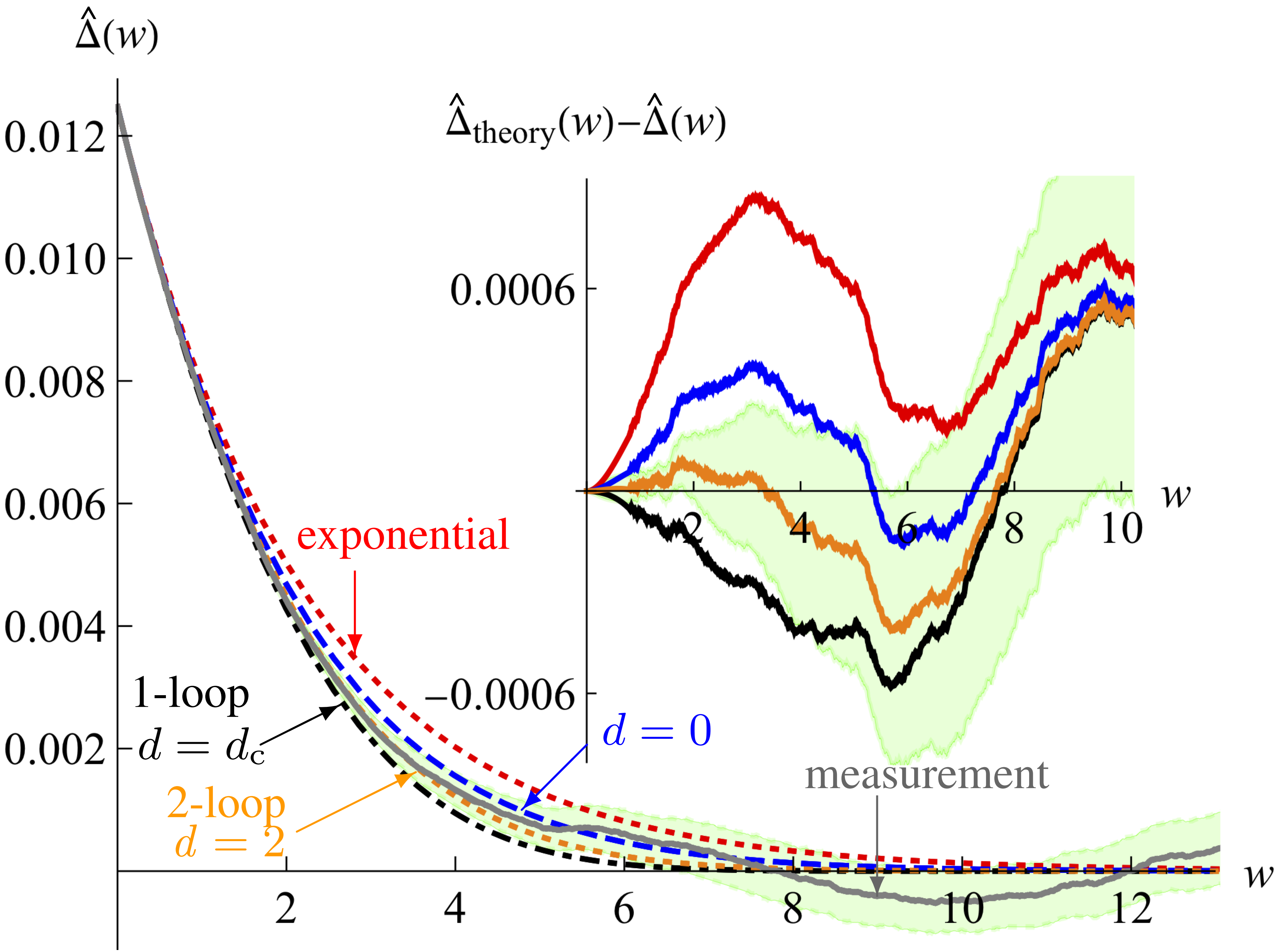
current in a pickup coil .... allows to construct :



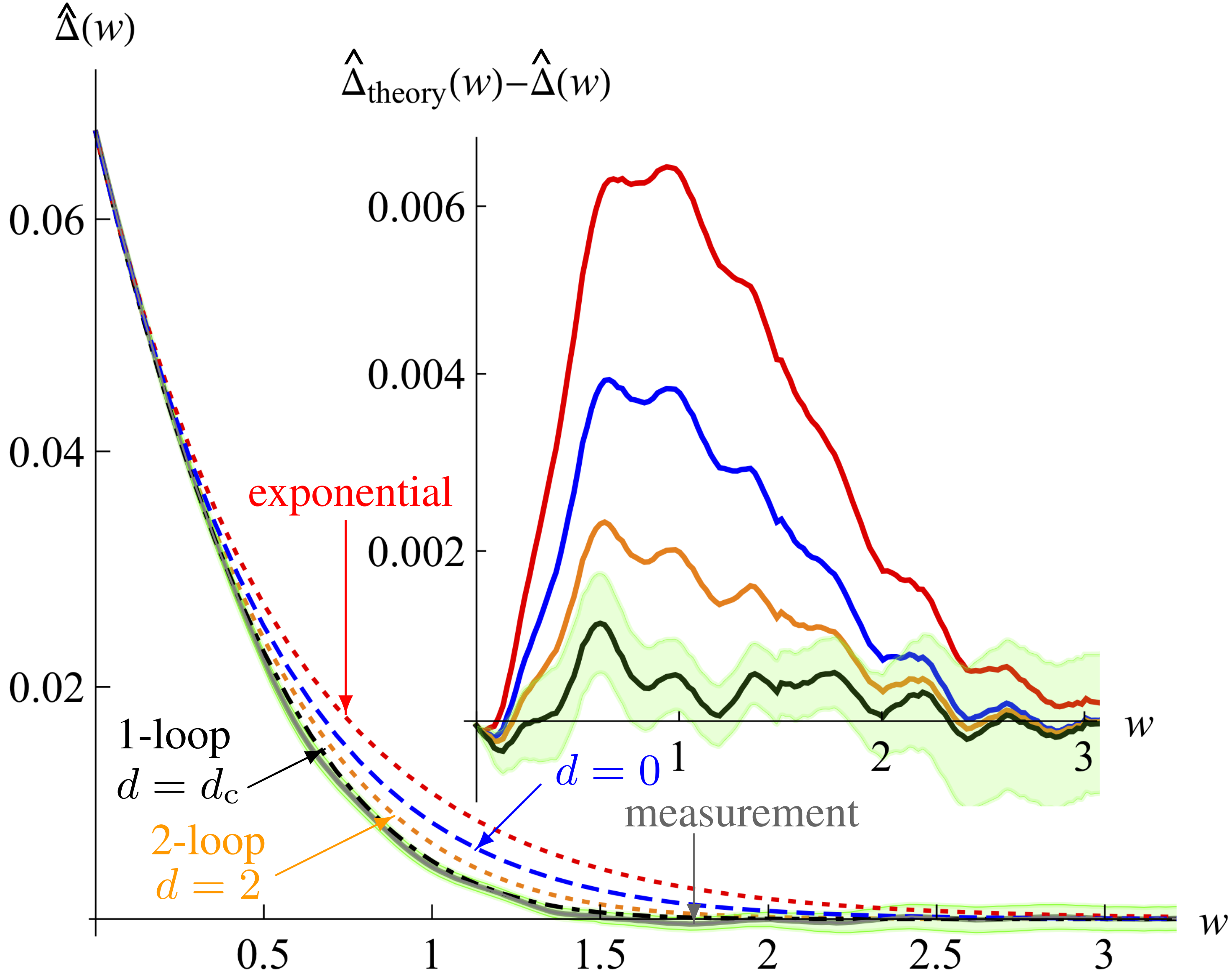
eliminate one unknown scale by the definition

$$\hat{\Delta}_v(w - w') := \overline{[w - u_w]\,[w' - u_{w'}]}^{c} = \frac{1}{m^4}\overline{F_w F_{w'}}^{c}$$

Magnetic domain walls SR elasticity $(d = 2)$



Magnetic domain walls $(d = 2)$ with LR elasticity



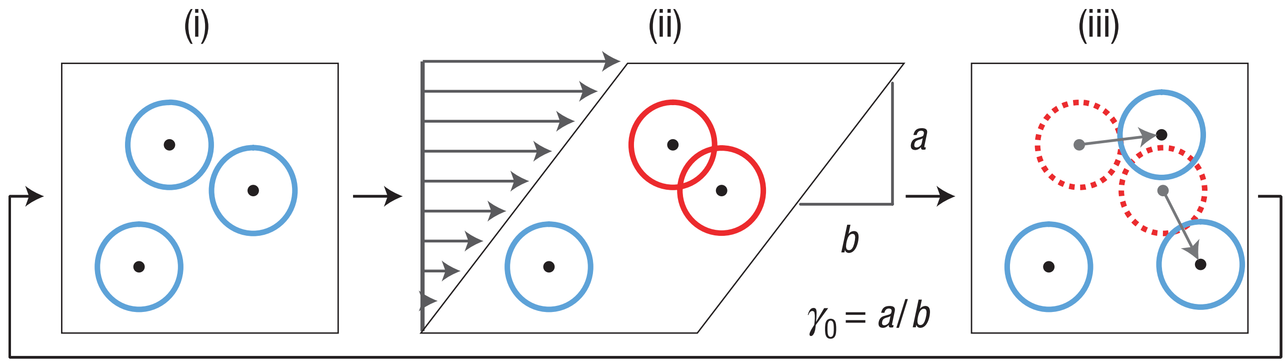
- **1-loop FRG gives fixed point.**
- **this is not ABBM disorder:** $\Delta(0) - \Delta(w) \neq \sigma|w|$
- **ABBM only gives short-scale behavior correctly**

Depinning
= Conserved Directed Percolation (CDP)
= Manna sandpiles

- CDP needs functional RG
- exponents at 2-loop order are highly nontrivial

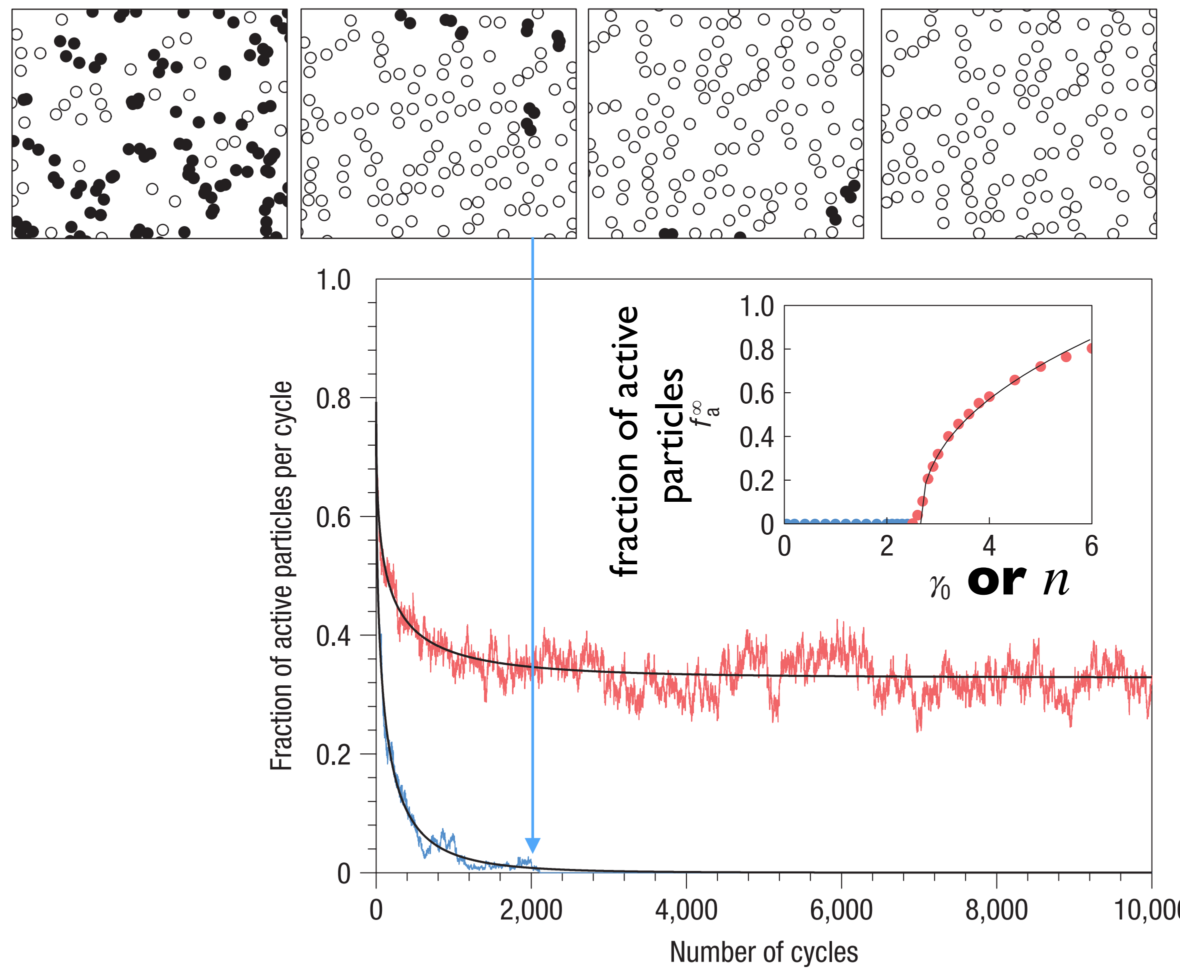
Periodically sheared colloids

$$\gamma(t) = \gamma_0 \cos(\omega t)$$



Corté, Chaikin, Gollub, Pine: Nature Physics 4 (2008) 420–424.

$\gamma_0 = 2$



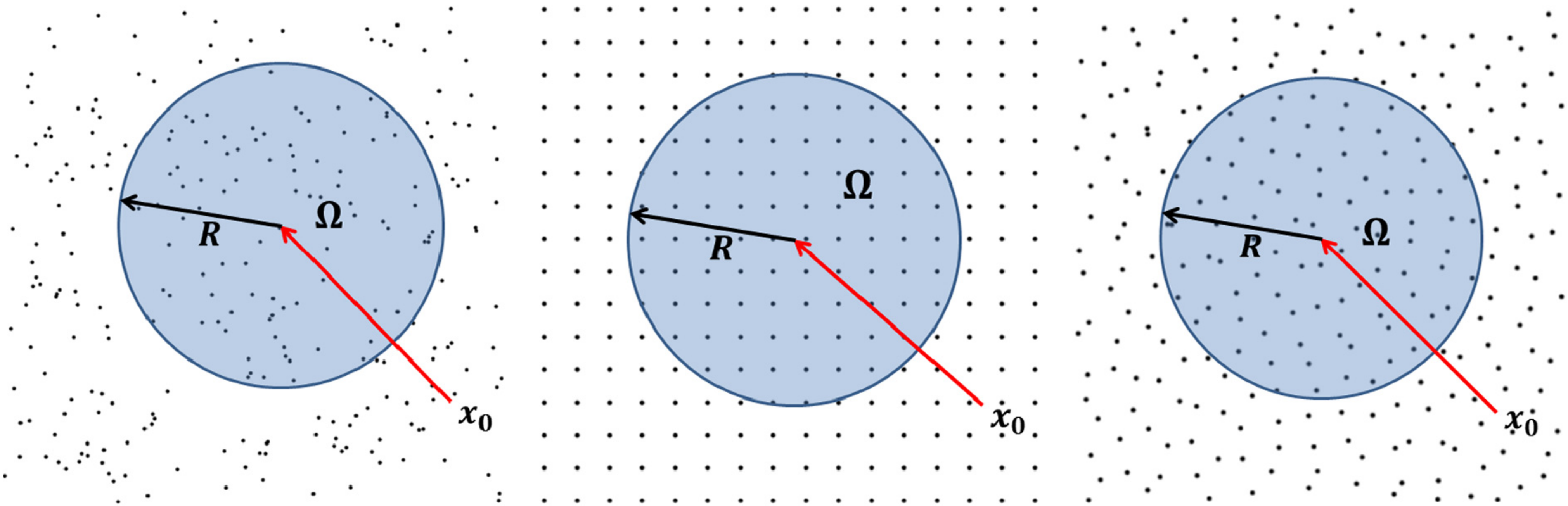
Phenomenology:

- phase transition
- *hyperuniform* at the transition

D. Hexner, D. Levine,
Phys. Rev. Lett. 118 (2017) 020601

Hyperuniformity

S. Torquato and F.H. Stillinger, PRE. E 68 (2003) 041113.
Phys. Rep. 754 (2018) 1-95



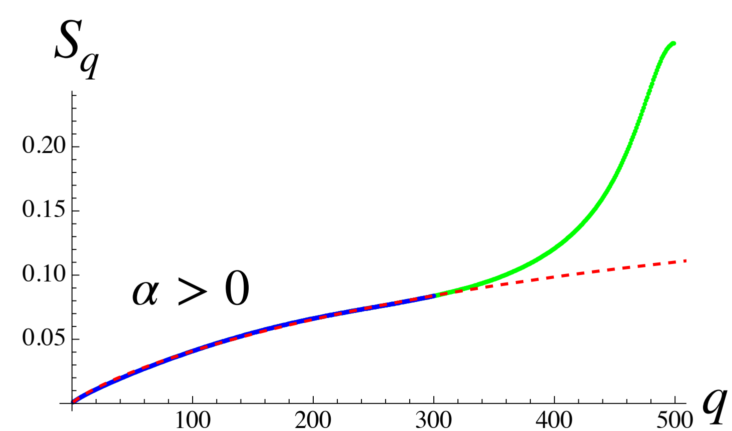
Poisson process — regular lattice — disordered hyperuniform (e.g. sheared colloid)

$$\langle N_R \rangle \sim \frac{N_{\rm tot}}{L^d} R^d, \quad \mathrm{var}(N_R) = \langle N_R^2 \rangle - \langle N_R \rangle^2 \sim R^\kappa, \quad d-1 \le \kappa \le d$$

In Fourier

$$S_q = \langle n_q n_{-q} \rangle \sim q^\alpha, \qquad \kappa + \alpha = d\,.$$

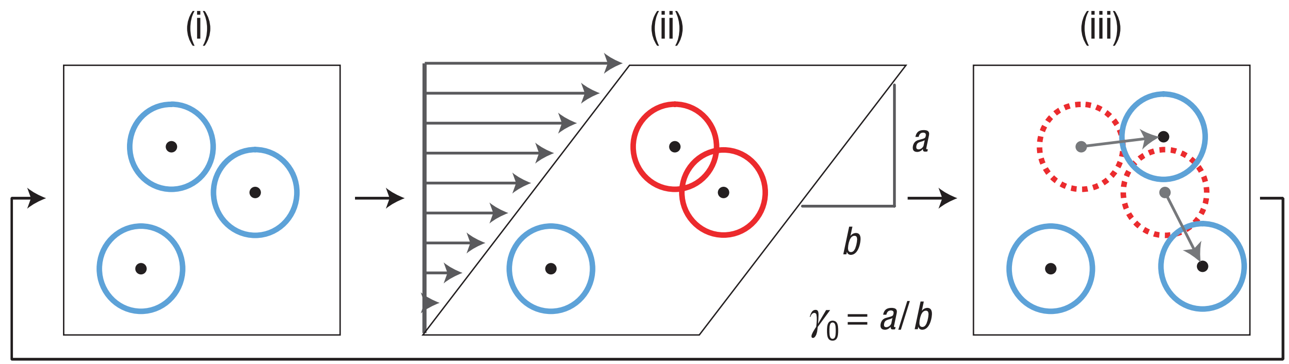
hyperuniformity exponent ↓ (α)

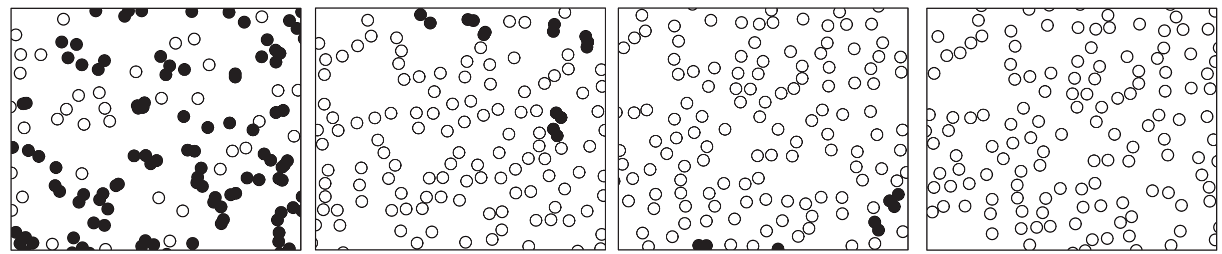


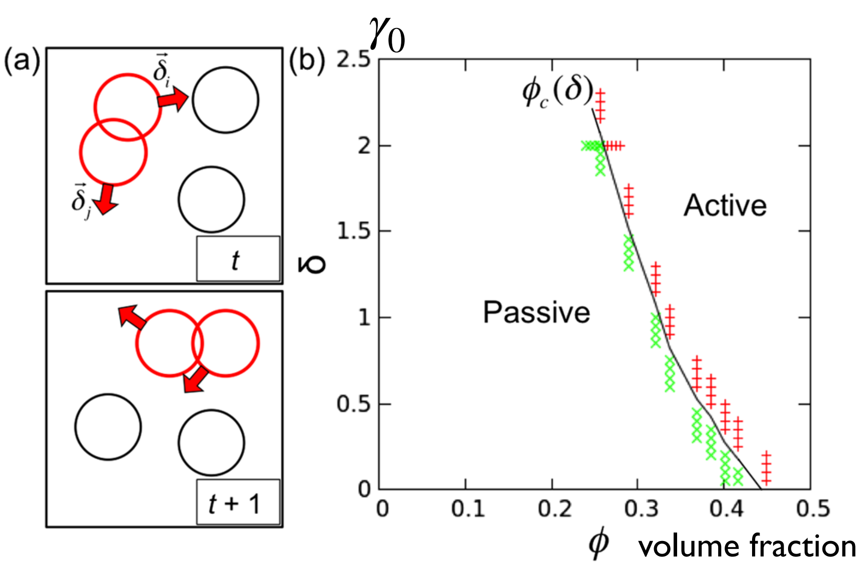
Random Organization Model (RO)

Corté, Chaikin, Gollub, Pine: Nature Physics 4 (2008) 420–424.

$$\gamma(t) = \gamma_0 \cos(\omega t)$$



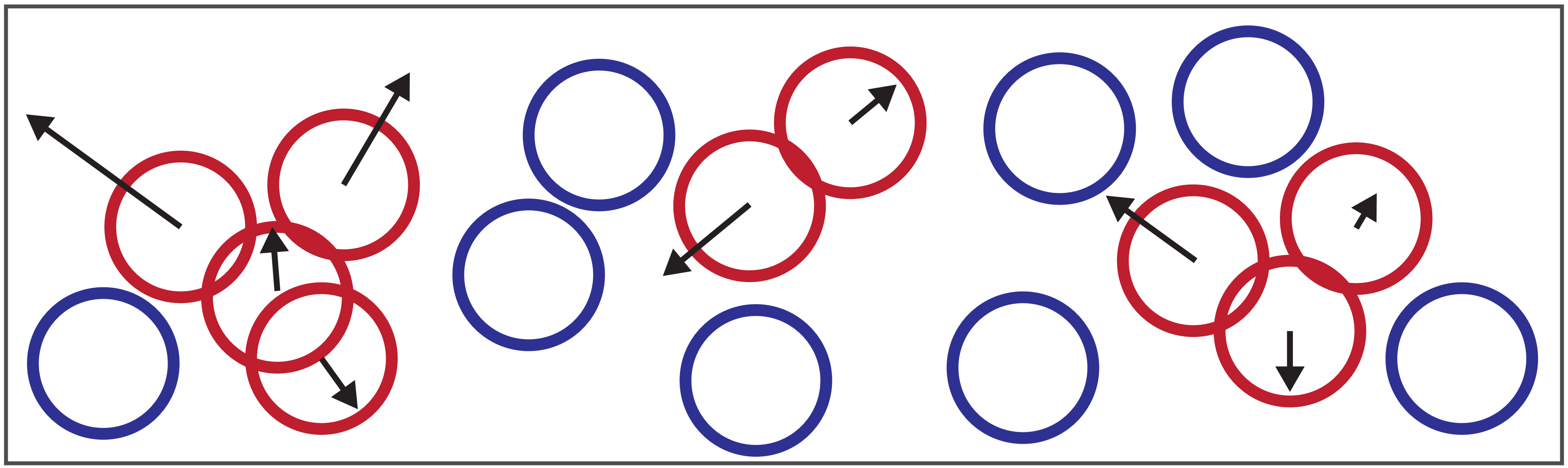




Tjhung, Berthier, PRL 114 (2015) 148301

- shear for one period
- mark particles which collided as active
- displace active particles in a random direction at distance $\sim \gamma_0$

Biased Random Organization Model (BRO)



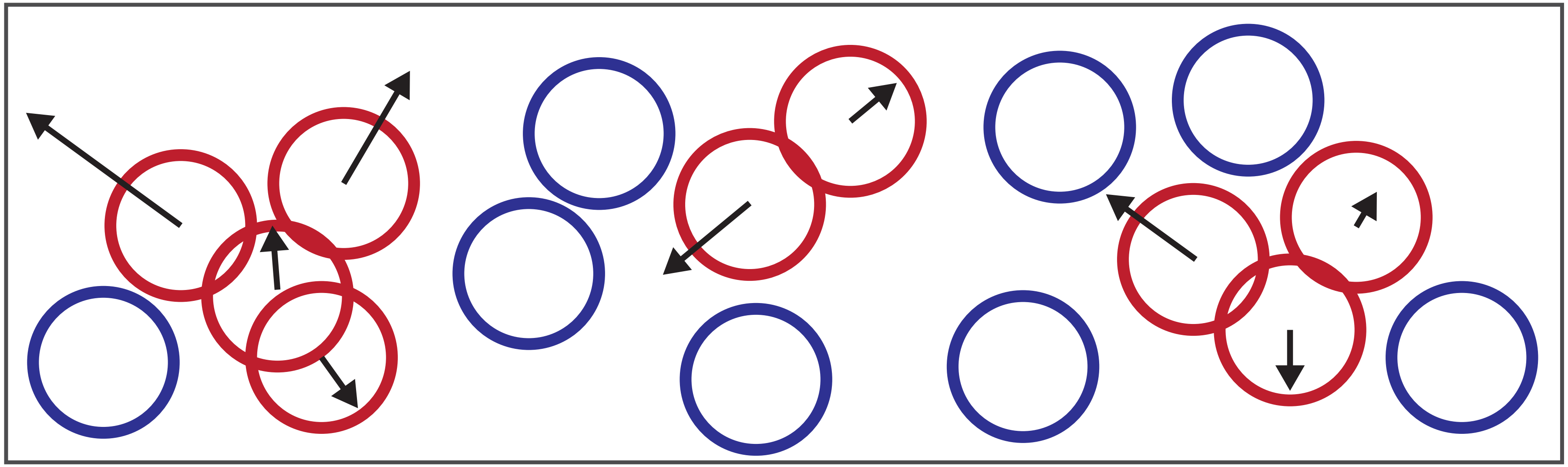
- move particles apart after collision
- seems to account for statistics of **sheared colloids** and **random closed packings**

S. Wilken, A.Z. Guo, D. Levine and P.M. Chaikin, PRL 131 (2023) 238202.

- probably wrong for dense packings

L. Berthier + K.J. Wiese, work in progress

Effective Theory for (Biased) Random Organization Model



Conserved Directed Percolation (CDP) effective equations

activity

$$\partial_t \rho(x,t) = [n(x,t) - \rho(x,t) - 1]\rho(x,t) + \nabla^2 \rho(x,t) + \sqrt{\rho(x,t)}\,\xi(x,t)$$

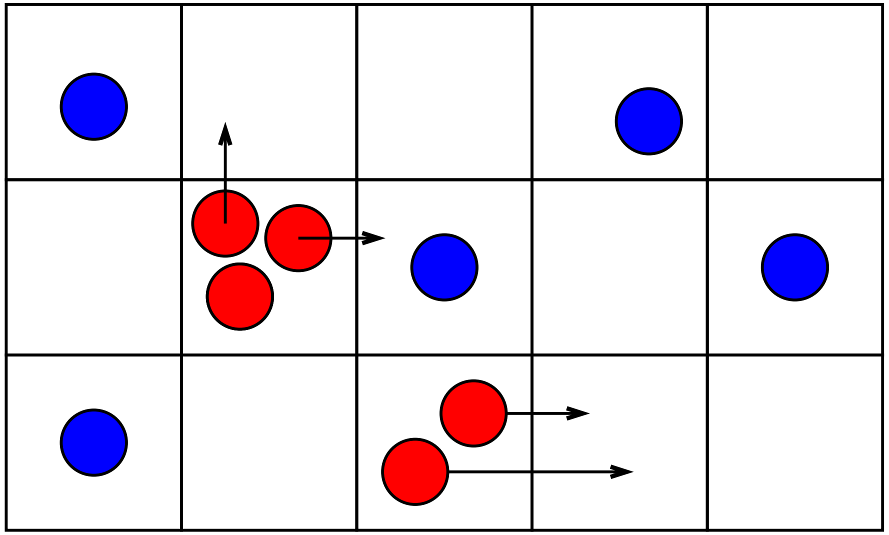
number of particles

$$\partial_t n(x,t) = \nabla^2 \rho(x,t)$$

variance of noise

noise

Relation to Manna sandpiles



Manna sandpile rule: If 2 or more grains are on a site, topple 2 to randomly chosen neighbors.

2 grains can end on same site.

CDP

activity → ρ

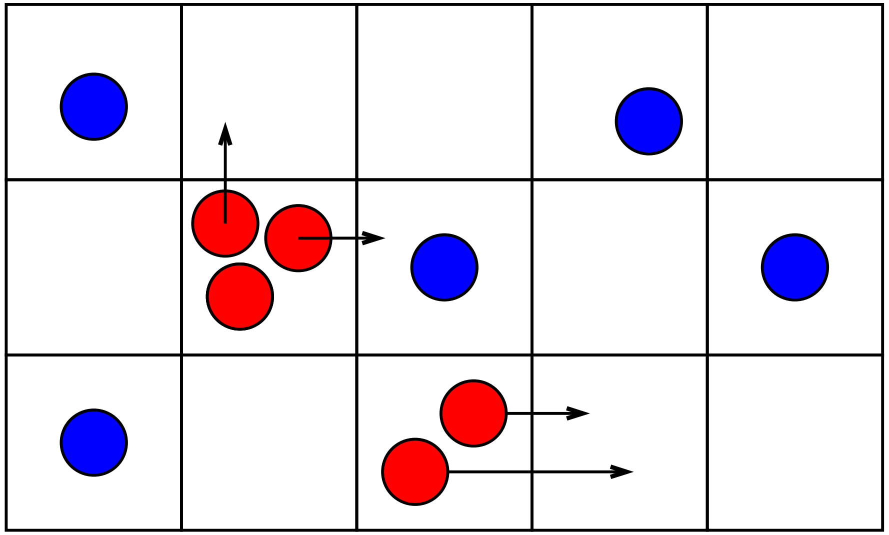
number of grains → n

$$\partial_t \rho(x,t) = [n(x,t) - \rho(x,t) - 1]\rho(x,t) + \nabla^2 \rho(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t)$$

$$\partial_t n(x,t) = \nabla^2 \rho(x,t)$$

noise → η

Manna sandpiles to CDP



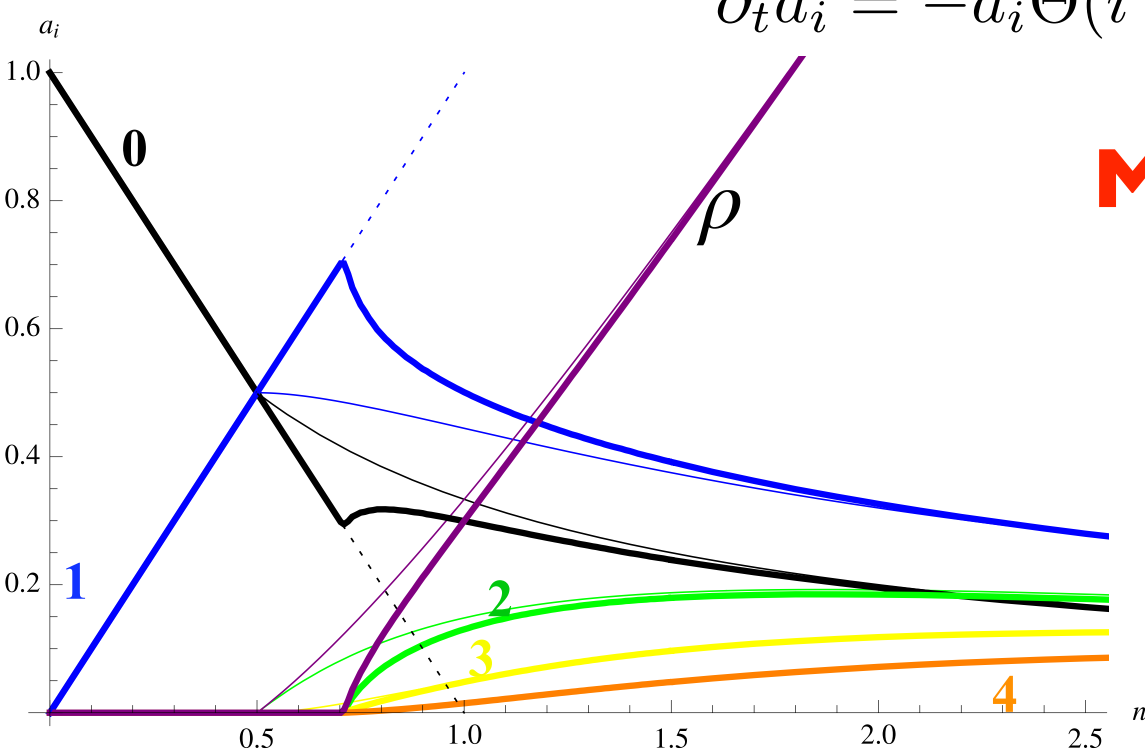
Manna sandpile rule: If 2 or more grains are on a site, topple them to randomly chosen neighbors.

2 grains can end up on same site.

KW, PRE 93 (2016) 042117

Mean-Field Equations

$$\partial_t a_i = -a_i \Theta(i \geq 2) + a_{i+2} + 2\left[\sum_{j\geq 2} a_j\right](a_{i-1} - a_i)$$



Mean-Field Solution

$$a_0 = \frac{1}{1+2n},$$

$$a_{i>0} = \frac{4n\left(\frac{2n-1}{2n+1}\right)^i}{4n^2-1}.$$

fraction of sites with i grains

Beyond Mean-Field

a_i = fraction of i times occupied sites

number of grains $n := \sum_i a_i \, i$

empty sites $e := a_0$ activity $\rho := \sum_{i \geq 2} a_i (i-1)$

sum rules $\sum_i a_i = 1$ and $n - \rho + e = 1$

CDP equations

activity → MF

$$\partial_t \rho(x,t) = [n(x,t) - \rho(x,t) - 1]\rho(x,t)$$

$$+ \nabla^2 \rho(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t)$$

number of grains → noise

$$\partial_t n(x,t) = \nabla^2 \rho(x,t)$$

CDP onto Depinning

activity →

$$\partial_t \rho(x,t) = [n(x,t) - \rho(x,t) - 1]\rho(x,t) + \nabla^2 \rho(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t)$$

number of grains →

$$\partial_t n(x,t) = \nabla^2 \rho(x,t)$$

← noise

Change variables: $n - \rho - 1 = -e$

$$\partial_t e(x,t) = -e(x,t)\rho(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t)$$
$$\partial_t \rho(x,t) = \nabla^2 \rho(x,t) + \partial_t e(x,t)$$

Identification with random manifold

PLD+KW: Phys. Rev. Lett. 114 (2014) 110601

$\rho(x,t) = \partial_t u(x,t)$ (the velocity of the interface)

$e(x,t) = \mathscr{F}(x,t)$ (the force acting on it)

Integrate second equation

$$\partial_t u(x,t) = \nabla^2 u(x,t) + \mathscr{F}(x,t) + f_0$$

Equation for force

$$\partial_t \mathscr{F}(x,t) = -\mathscr{F}(x,t)\partial_t u(x,t) + \sqrt{\partial_t u(x,t)}\,\eta(x,t)$$

Parameterize F as a function of u instead of t

$$\begin{aligned}\partial_t \mathscr{F}(x,t) &\to \partial_t F\big(x, u(x,t)\big)\\ &= \partial_u F(x,u(x,t))\partial_t u(x,t)\\ &= -F(x,u(x,t))\partial_t u(x,t) + \sqrt{\partial_t u(x,t)}\,\eta(x,t)\end{aligned}$$

F as a function of u: Ornstein-Uhlenbeck process

$$\partial_u F(x,u) = -F(x,u) + \xi(x,u),$$

(compare variances: $\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \to \mathrm{d}u$)

$$\langle \xi(x,u)\xi(x',u')\rangle = \delta^d(x-x')\delta(u-u')$$

Correlations of F are short-ranged

$$\langle F(x,u)F(x',u')\rangle = \delta^d(x-x')\mathrm{e}^{-|u-u'|}\,.$$

CDP equivalent to depinning!

CDP equivalent to depinning with SR correlations

activity

$$\partial_t \rho(x,t) = [n(x,t) - \rho(x,t) - 1]\rho(x,t) + \nabla^2 \rho(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t)$$

number of grains

$$\partial_t n(x,t) = \nabla^2 \rho(x,t)$$

dynamic noise

Identification with random manifold

$\rho(x,t) = \partial_t u(x,t)$ (the velocity of the interface)

$e(x,t) = \mathscr{F}(x,t)$ (the force acting on it)

$$\partial_t u(x,t) = \nabla^2 u(x,t) + F(x,u(x,t)) + f_0$$

$$\langle F(x,u)F(x',u')\rangle = \delta^d(x-x')\mathrm{e}^{-|u-u'|}\,.$$

quenched noise

Density fluctuations in the C-DP field theory

PRL 133 (2024) 067103
arXiv:2401.09123

$$\partial_t n(x,t) = \nabla^2 \rho(x,t),\ \ \rho(x,t) \equiv \partial_t u(x,t)$$

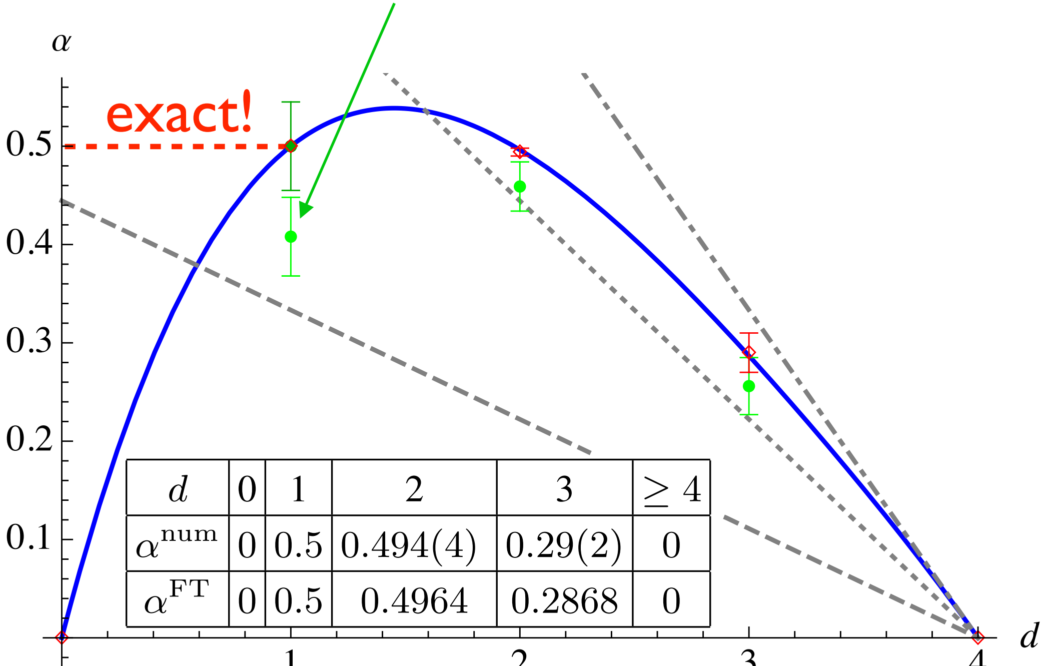
Integrate over t: $n(x,t) = \nabla^2 u(x,t) + n_0$

$$\langle n(x,t) n(x',t)\rangle^{\rm c} = -(\nabla^2)^2 \frac{1}{2} \left\langle [u(x,t) - u(y,t)]^2 \right\rangle$$

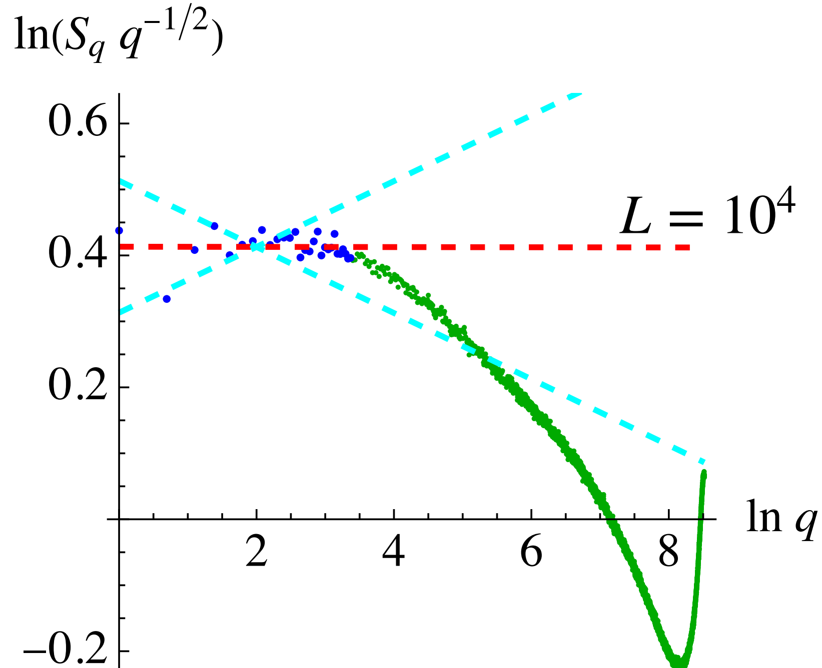
Structure factor is **hyperuniform**: $S_q = \langle n_q n_{-q} \rangle \sim q^{\alpha}$

$\alpha = 4 - d - 2\zeta,$ where $\zeta =$ roughness at depinning

M. Henkel, H. Hinrichsen and S. Lubeck, Non-Equilibrium Phase Transitions (2008)

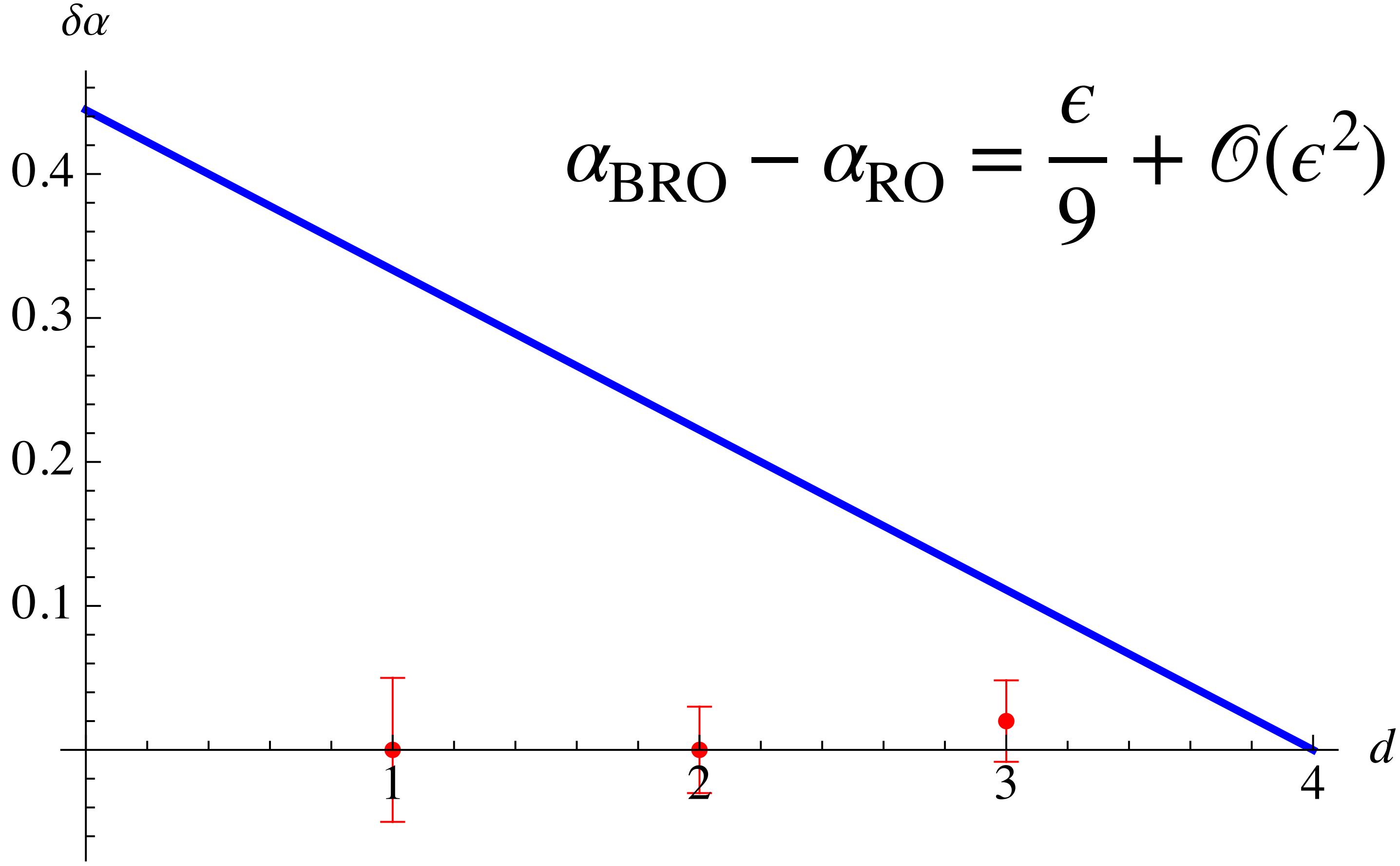


d	0	1	2	3	≥ 4
$\alpha^{\rm num}$	0	0.5	0.494(4)	0.29(2)	0
$\alpha^{\rm FT}$	0	0.5	0.4964	0.2868	0



d=1: $\zeta = 5/4$:Shapira, KW: J. Stat. Mech. 2023 (2023) 063202
3loop: Semeikin+KW, Phys. Rev. B 109 (2024) 134203

BRO and RO have different hyperuniformity?



$$\alpha_{\mathrm{BRO}} - \alpha_{\mathrm{RO}} = \frac{\epsilon}{9} + \mathcal{O}(\epsilon^2)\,.$$

[1] X. Ma, J. Pausch and M.E. Cates, Theory of hyperuniformity at the absorbing state transition 2023, arXiv:2310.17391.
[2] X. Ma, J. Pausch, G. Pruessner and M.E. Cates, Hyperuniformity at the absorbing state transition: Perturbative RG for Random Organization 2025, arXiv:2507.07793.

Conclusions

- hyperuniformity in the CDP class is linked to depinning: $\alpha = 4 - d - 2\zeta$
- many systems belong to the CDP class:
  - Manna sandpiles
  - epidemic models: $H + S \to 2S$, $S \to H$
  - sheared colloids
  - random packings
  - your favorite system?
- DP is harder to achieve than CDP, maybe impossible in presence of a conserved density
- closed random packings in CDP?!

PRL 133 (2024) 067103, arXiv:2401.09123

Theory and Experiments for Disordered Elastic Manifolds, Depinning, Avalanches, and Sandpiles

Review

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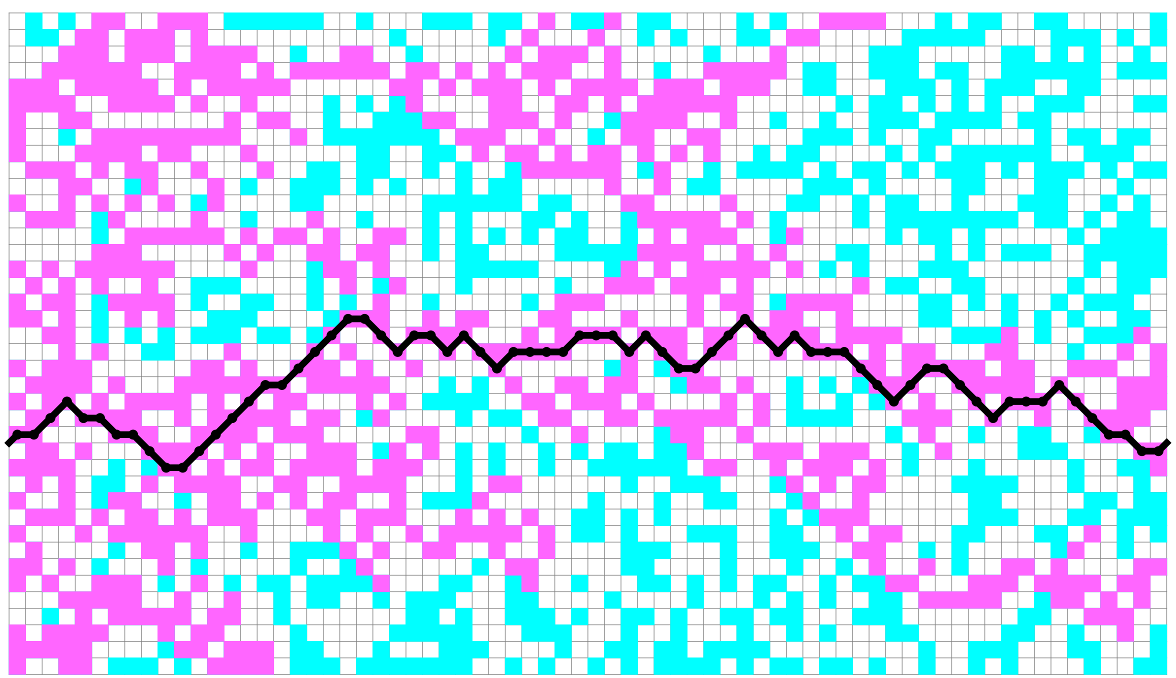
1 September 2021 – masterENS.tex – REVISION 1.1083

Rep. Prog. Phys. 85 (2022) 086502 (133pp)
arXiv:2102.01215.

http://www.phys.ens.fr/~wiese/

Abstract. Domain walls in magnets, vortex lattices in superconductors, contact lines at depinning, and many other systems can be modeled as an elastic system subject to quenched disorder. The ensuing field theory posesses a well-controlled perturbative expansion around its upper critical dimension. Contrary to standard field theory, the renormalization group flow involves a function, the disorder correlator $\Delta(w)$, and is therefore termed the functional renormalization group (FRG). $\Delta(w)$ is a physical observable, the auto-correlation function of the center of mass of the elastic manifold. In this review, we give a pedagogical introduction into its phenomenology and techniques. This allows us to treat both equilibrium (statics), and depinning (dynamics). Building on these techniques, avalanche observables are accessible: distributions of size, duration, and velocity, as well as the spatial and temporal shape. Various equivalences between disordered elastic manifolds, and sandpile models exist: an elastic string driven at a point and the Oslo model; disordered elastic manifolds and Manna sandpiles; charge density waves and Abelian sandpiles or loop-erased random walks. Each of the mappings between these systems requires specific techniques, which we develop, including modeling of discrete stochastic systems via coarse-grained stochastic equations of motion, super-symmetry techniques, and cellular automata. Stronger than quadratic nearest-neighbor interactions lead to directed percolation, and non-linear surface growth with additional KPZ terms. On the other hand, KPZ without disorder can be mapped back to disordered elastic manifolds, either on the directed polymer for its steady state, or a single particle for its decay. Other topics covered are the relation between functional RG and replica symmetry breaking, and random field magnets. Emphasis is given to numerical and experimental tests of the theory.



Anisotropic depinning with its relation to directed percolation, explained in section 5.7.

today:PRL 133 (2024) 0671
arXiv:2401.09123.