

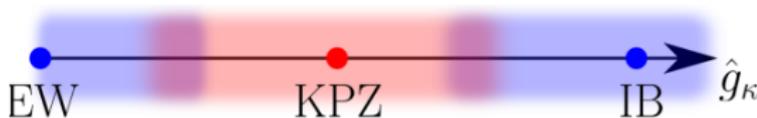
# Inhomogeneous Random Systems

27-28 January 2026



## Non-equilibrium and disordered interfaces

The non-perturbative sides of the Kardar-Parisi-Zhang equation



Léonie Canet  
Université Grenoble Alpes

# Outline

- 1** The strong-coupling Kardar-Parisi-Zhang fixed point in dimension  $d > 1$
- 2** The inviscid-Burgers fixed point

# Acknowledgments

## Collaborators



B. Delamotte  
LPTMC Paris



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Montevideo



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CEA Paris



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ENS Paris

## PhD students and post-docs



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M. Tarpin



A. Gorbunova



C. Fontaine



F. Vercesi



L. Gosteva

## References

- LC, B. Delamotte, H. Chaté, N. Wschebor, PRL **104** (2010), PRE **84** (2011)  
C. Fontaine, F. Vercesi, M. Brachet, LC, PRL **131** (2023)  
L. Gosteva, M. Tarpin, N. Wschebor, LC, PRE **110** (2024)  
L. Gosteva, N. Wschebor, LC, J. Stat. Mech. **2025** 114002  
LC, J. Stat. Mech. **2025** 124003

# The Kardar-Parisi-Zhang equation

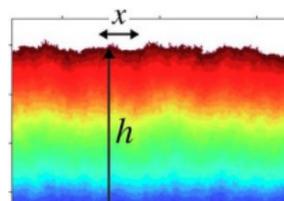
## ► KPZ equation for stochastically growing interfaces

Kardar, Parisi, Zhang, PRL 56 (1986)

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \sqrt{D} \eta$$

$\eta$ : Gaussian random noise of variance

$$\langle \eta(t, \mathbf{x}) \eta(t', \mathbf{x}') \rangle = 2\delta(t - t') \delta^d(\mathbf{x} - \mathbf{x}')$$



Takeuchi *et al*, Sci. Rep. 1 (2011)

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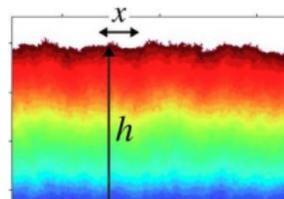
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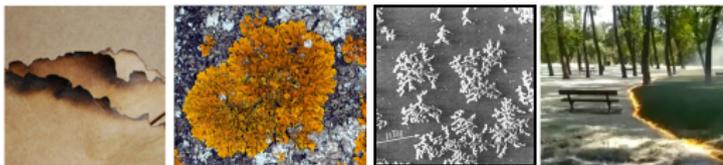
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## ► kinetic roughening as a non-equilibrium critical phenomenon

- generic **scale-invariance**
- **universality**

Halpin-Healy, Zhang, Phys. Rep. 254 (1995)

Krug, Adv. Phys. 46 (1997)



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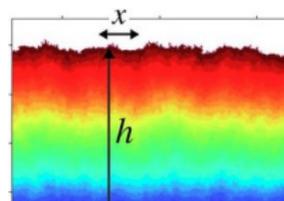
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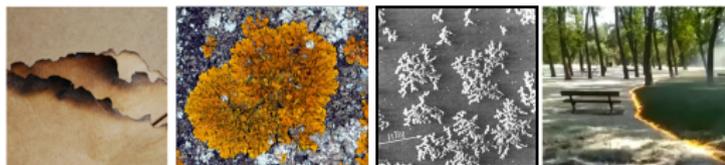
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Krug, Adv. Phys. **46** (1997)



- ▶ **correlation function** takes Family-Vicsek scaling form

$$C(t, \mathbf{x}) = \langle (h(t, \mathbf{x}) - h(0, 0))^2 \rangle = |\mathbf{x}|^{2\chi} \mathcal{F}(t/|\mathbf{x}|^z) \quad \begin{cases} \chi: \text{roughness exponent} \\ z: \text{dynamical exponent} \end{cases}$$

# The Kardar-Parisi-Zhang equation

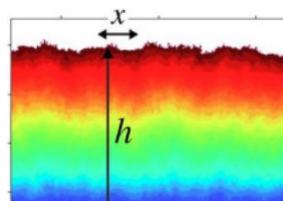
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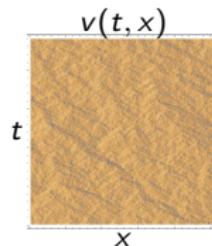
Takeuchi *et al* Sci. Rep. 1 (2011)

⇒ **exact mapping to**

- ▶ **Burgers equation** for randomly stirred fluids Burgers (1948)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \nabla^2 \mathbf{v} + \sqrt{D} \nabla \eta$$

$$\text{for } \mathbf{v} = -\lambda \nabla h \text{ with } \nabla \times \mathbf{v} = 0$$



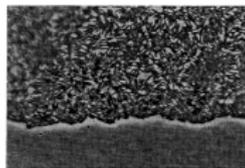
Brachet (2021)

- ▶ **Directed Polymers** in random media and stochastic heat equation

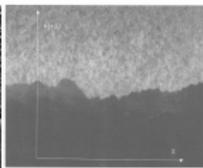
# The Kardar-Parisi-Zhang equation

## An extremely large universality class

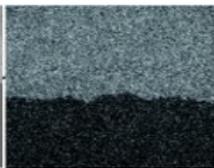
### ► KPZ in stochastic growth processes



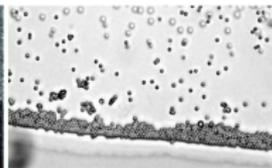
**bacteriae**  
PRL (1997)



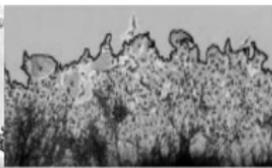
**combustion**  
JPSJ (1997)



**liquid crystals**  
PRL (2010)



**deposition**  
Nature (2011)

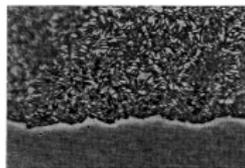


**cancer cells**  
PRE (2012)

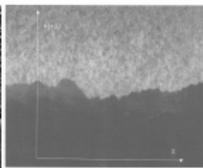
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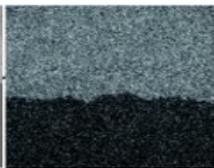
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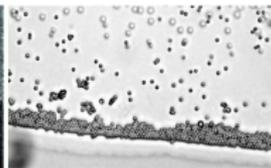
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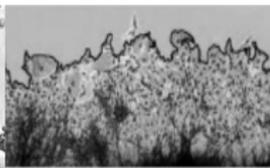
combustion  
JPSJ (1997)



liquid crystals  
PRL (2010)



deposition  
Nature (2011)



cancer cells  
PRE (2012)

### ► KPZ in quantum systems

#### ■ integrable Heisenberg spin chains

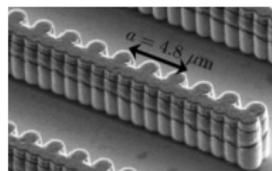
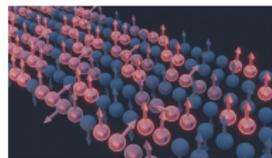
- ▷ solid-state magnets *Nature Phys.* 17 (2021)
- ▷ ultra-cold gases *Science* 316 (2022)

#### ■ driven-dissipative Bose-Einstein condensates

- ▷ exciton-polaritons *Nature* 608 (2022)

#### ■ entanglement growth *PRX* 7 (2017)

#### ■ Anderson localisation *PRL* 132 (2024)



# The 1D Kardar-Parisi-Zhang equation: Scaling ...

▶ space-time correlation function  $C(t, \mathbf{x}) = |\mathbf{x}|^{2\chi} \mathcal{F}(t/|\mathbf{x}|^z)$

▷ linear growth (EW):  $\lambda = 0$

Edwards, Wilkinson, Proc.R.Soc.Lond. A (1982)

$$\chi = 1/2, \quad z = 2$$

▷ nonlinear growth (KPZ)

Kardar, Parisi, Zhang, PRL 56 (1986)

$$\chi = 1/2, \quad z = 3/2$$

# The 1D Kardar-Parisi-Zhang equation: ... and beyond

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$$\chi = 1/2, \quad z = 2$$

$$\chi = 1/2, \quad z = 3/2$$

## ★ universal distribution of height fluctuations



### ■ curved geometry – droplet (GUE-TW)

Sasamoto, Spohn, PRL 104 (2010)

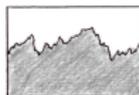
Amir, Corwin and Quastel, Commun. Pure Appl. Math. 64 (2011)

Calabrese, Le Doussal, Rosso, EPL 90 (2010)



### ■ flat geometry (GOE-TW)

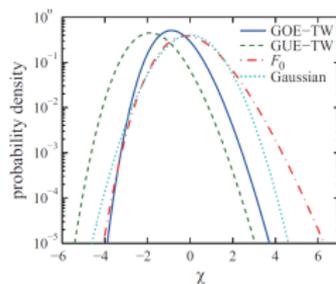
Calabrese, Le Doussal, PRL 106 (2011)



### ■ Brownian geometry (Baik-Rains $F_0$ )

Imamura, Sasamoto, PRL (2012)

Borodin, Corwin, Ferrari, Vetö, Math. Phys. Ann. Geom. 18 (2015)



## ★ universal scaling functions $\mathcal{F}$ : in terms of Airy processes

Prahöfer, Spohn, J. Stat. Phys. (2004), Sasamoto, J. Phys. A (2005), Imamura, Sasamoto, PRL (2012)

# The 1D Kardar-Parisi-Zhang equation: Scaling and beyond

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$$\chi = 1/2, \quad z = 3/2$$

▶ what about higher dimensions ?

▶ still holds surprises even in  $d = 1$  !

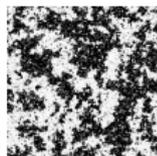
# Outline

- 1** The strong-coupling Kardar-Parisi-Zhang fixed point in dimension  $d > 1$
- 2** The inviscid-Burgers fixed point

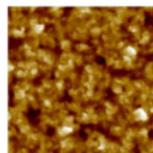
# Critical phenomena and Renormalisation Group (RG)

► kinetic roughening is a **non-equilibrium critical phenomena**

- scale invariance, self-similarity
- universality
- anomalous critical exponents



Ising second order  
phase transition



2D KPZ  
interface

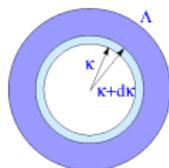
Halpin-Healy

Palasantzas

EPL 105 (2014)

⇒ **criticality arises from fluctuations at all scales ...**

► **Wilson's RG** pivotal to understand critical phenomena



- progressive integration of fluctuation modes
- build effective theory

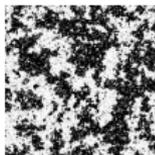
Wilson, Kogut, Phys. Rep. C 12 (1974)

scale invariance  $\iff$  fixed point of the RG

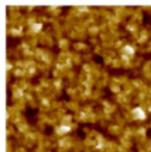
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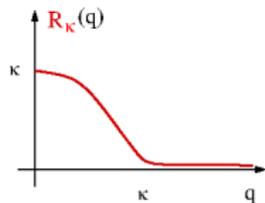
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⇒ **criticality arises from fluctuations at all scales ...**

► **Functional non-perturbative RG** based on Wilson's RG ideas



■ Effective average action  $\Gamma_\kappa$  instead of effective action  $\mathcal{S}_\kappa$

■ Exact RG equation for  $\Gamma_\kappa$

Wetterich, Phys. Lett. B **301** (1993), Dupuis, *et al*, Phys. Rep. 910 (2021)

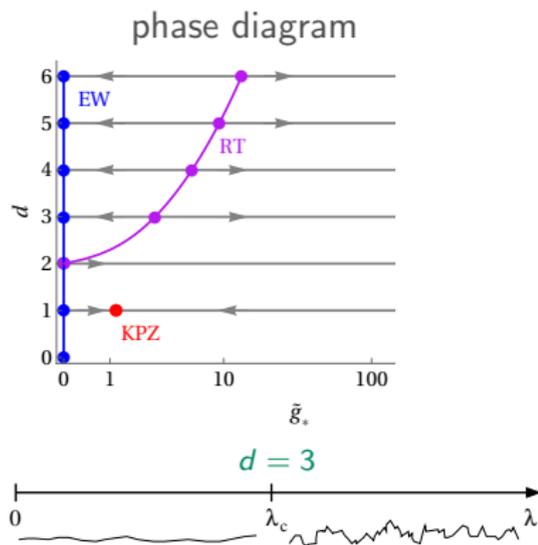
$$\partial_\kappa \Gamma_\kappa = \frac{1}{2} \text{Tr} \int_{\mathbf{q}} \partial_\kappa R_\kappa(\mathbf{q}) \left[ \Gamma_\kappa^{(2)} + R_\kappa \right]^{-1}(-\mathbf{q})$$

# The non-perturbative side of the KPZ equation (I)

- ▶ one coupling constant  $g = \lambda^2 D / \nu^3$

$$\partial_t h = \nabla^2 h + \frac{\sqrt{g}}{2} (\nabla h)^2 + \eta$$

- ▶ Perturbative Renormalisation Group



- ▶ at one-loop order in  $d = 1$

**KPZ rough phase** in  $d = 1$

$$z = 3/2 \text{ and } \chi = 1/2$$

Forster, Nelson, Stephen, PRL **36** (1977)

Kardar, Parisi, Zhang, PRL **56** (1986)

- ▶ resummed to all orders in  $d = 2 + \varepsilon$

**Roughening transition** in  $d > 2$

$$z = 2 \text{ and } \chi = 0 \text{ for all } \varepsilon$$

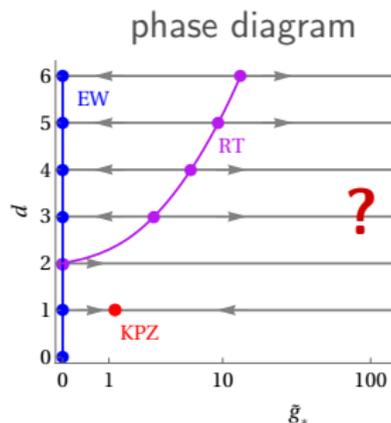
Wiese, PRE **56** (1997), J. Stat. Phys. **93** (1998)

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Roughening transition in  $d > 2$

$z = 2$  and  $\chi = 0$  for all  $\varepsilon$

Wiese, PRE **56** (1997), J. Stat. Phys. **93** (1998)

but does not find a KPZ strong-coupling fixed point in  $d > 1$

# The KPZ fixed point from Functional Renormalisation Group (FRG)

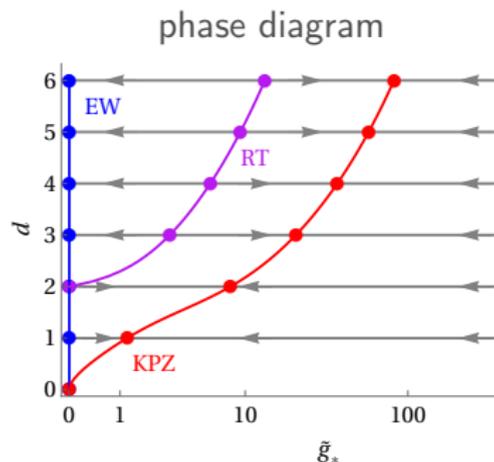
► FRG with simplest approximation:  $\nu \longrightarrow \nu_{\kappa}$ ,  $D \longrightarrow D_{\kappa}$

▷ one effective coupling  $\hat{g}_{\kappa} = \kappa^{d-2} \lambda^2 D_{\kappa} / \nu_{\kappa}^3$

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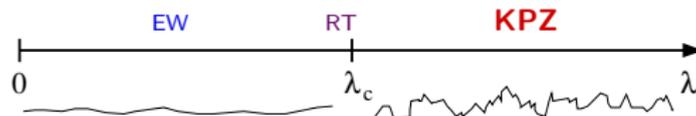


KPZ fixed point in all  $d$ !

LC, Chaté, Delamotte, Wschebor, PRL **104** (2010)

T. Kloss, LC, N. Wschebor, PRE **86** (2012)

$d = 3$

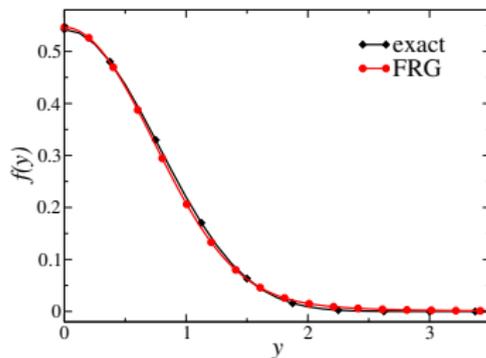
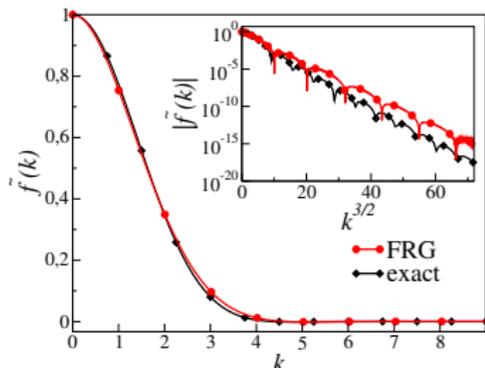


# The KPZ fixed point from Functional Renormalisation Group (FRG)

- ▶ FRG with functional approximation:  $\nu \longrightarrow \nu_\kappa(\omega, \mathbf{p})$ ,  $D \longrightarrow D_\kappa(\omega, \mathbf{p})$
- ▷ correlation function : shows generic scaling

$$C_\kappa(\omega, \mathbf{p}) = \frac{2D_\kappa(\omega, \mathbf{p})}{\omega^2 + \mathbf{p}^4 \nu_\kappa^2(\omega, \mathbf{p})} \xrightarrow{\kappa \rightarrow 0} \frac{1}{\mathbf{p}^{d+2+\chi}} \tilde{f}(\omega/\mathbf{p}^z)$$

universal scaling function  $\tilde{f}$  in  $d = 1$



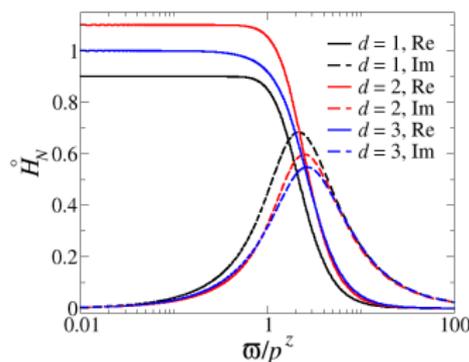
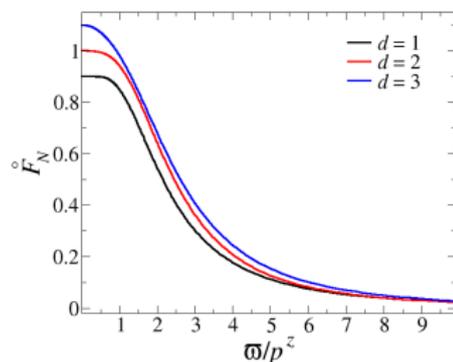
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universal scaling functions for correlation  $\mathring{F}$  and response  $\mathring{H}$  in  $d > 1$



# Comparison with numerics

## Other results for non-integrable cases

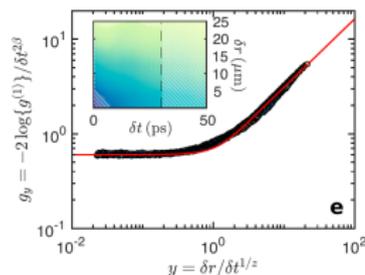
### ► universal amplitude ratio in $d = 2$ from large-scale simulations

■ FRG:  $R = 0.940$  Kloss, LC, Wschebor, PRE 86 (2012)

■ numerics:  $R = 0.944 \pm 0.031$  Halpin-Healy, PRE 88 (2013)

### ► universal scaling function for 2D exciton-polariton condensates

collaboration C2N – LPMMC, *in prep.* (2026)



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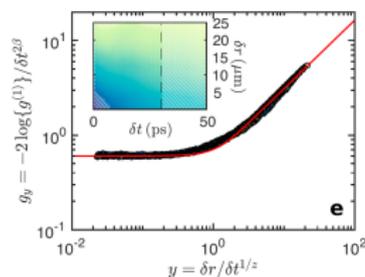
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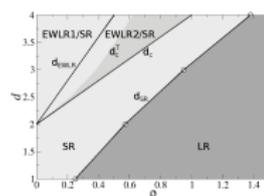
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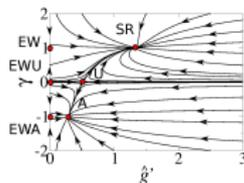
### ► KPZ equation with correlated noise

- long-range spatial noise Kloss, LC, Delamotte, Wschebor, PRE **89** (2014)
- short-range spatial noise Mathey, Agoritsas, Kloss, Lecomte, LC, PRE **95** (2017)
- long-range temporal noise Squizzato, LC, PRE **100** (2019)



### ► KPZ equation with anisotropy

Kloss, LC, Wschebor, PRE **90** (2014)



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# The 1D KPZ – Burgers equation in the inviscid limit

## The Galerkin-truncated Burgers equation: Crossover from inviscid-thermalised to Kardar-Parisi-Zhang scaling

C. Cartes<sup>1</sup>, E. Tirapegui<sup>2</sup>, R. Pandit<sup>3</sup> and M. Brachet<sup>4</sup>

Phil. Trans. A 380 (2022)

## Family-Vicsek Scaling of Roughness Growth in a Strongly Interacting Bose Gas

Kazuya Fujimoto<sup>1,2</sup>, Ryusuke Hamazaki<sup>3,4</sup> and Yuki Kawaguchi<sup>2</sup>

Phys. Rev. Lett. 124 (2020)

(c)

| Model                    | $\alpha$          | $\beta$           | $z$             |
|--------------------------|-------------------|-------------------|-----------------|
| KPZ                      | 1/2               | 1/3               | 3/2             |
| EW                       | 1/2               | 1/4               | 2               |
| BHM ( $\nu \gg 1$ )      | $0.517 \pm 0.030$ | $0.255 \pm 0.012$ | $2.07 \pm 0.20$ |
| BHM ( $\nu \simeq 1/2$ ) | $0.500 \pm 0.003$ | $0.489 \pm 0.004$ | $1.00 \pm 0.01$ |

this Letter

## Anomalous ballistic scaling in the tensionless or inviscid Kardar-Parisi-Zhang equation

Enrique Rodríguez-Fernández<sup>1,\*</sup>, Silvia N. Santalla<sup>2,†</sup>, Mario Castro<sup>3,‡</sup> and Rodolfo Cuerno<sup>1,§</sup>

Phys. Rev. E 106 (2022)

observation of an unpredicted scaling  $z = 1$  in the limit  $\nu \rightarrow 0$

- ▷ also found in Burgers-Hopf Majda, Timofeyev, PNAS 96 (2000)
- ▷ also predicted in  $d \rightarrow \infty$ ,  $\text{Re} \rightarrow \infty$  Bouchaud, Mézard, Parisi, PRE 52 (1995)

# Results from numerical simulations

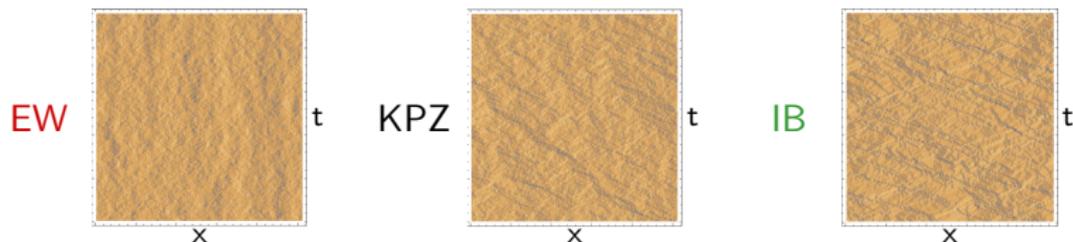
## ► simulation of stochastic 1D Burgers equation

Cartes, Tirapegui, Pandit, Brachet, Phil. Trans. A 380 (2022)

$$\partial_t v + \lambda v \partial_x v = \nu \partial_x^2 v + \sqrt{D} \partial_x \eta$$

→ Galerkin-truncation preserving all the symmetries  $\implies \chi = 1/2$

## ► observation of three scaling regimes



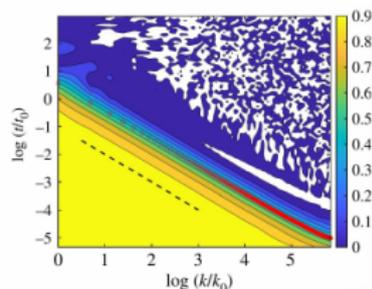
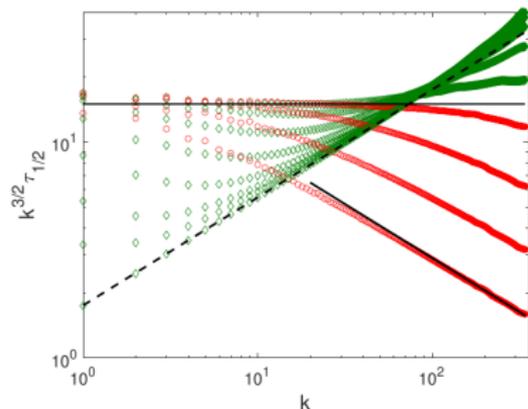
# Results from numerical simulations

- ▶ space-time correlation function  $C(t, k)$ :

$$\tau_{1/2}(k) \text{ such that } C(\tau_{1/2}(k), k) = \frac{1}{2} C(0, k)$$

$$\text{scaling regime: } \tau_{1/2} \sim k^{-z}$$

compensated half decay frequency

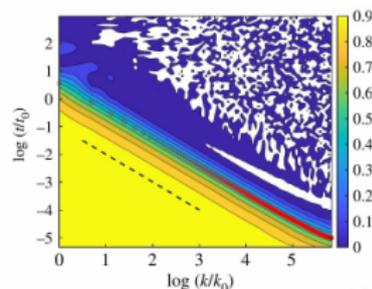


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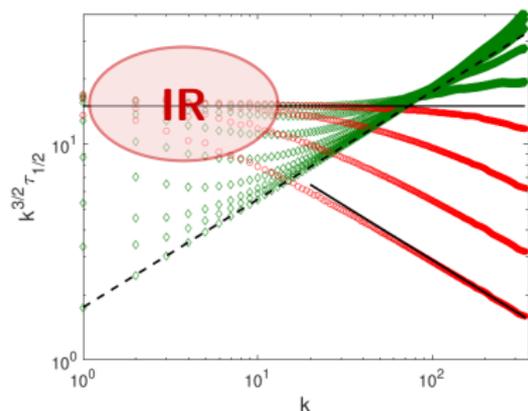
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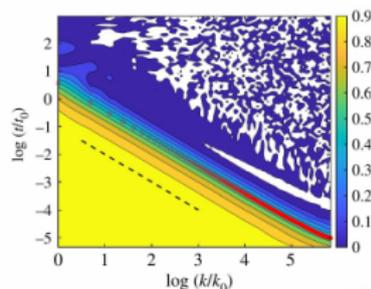
Kardar-Parisi-Zhang:  $z = 3/2$

# Results from numerical simulations

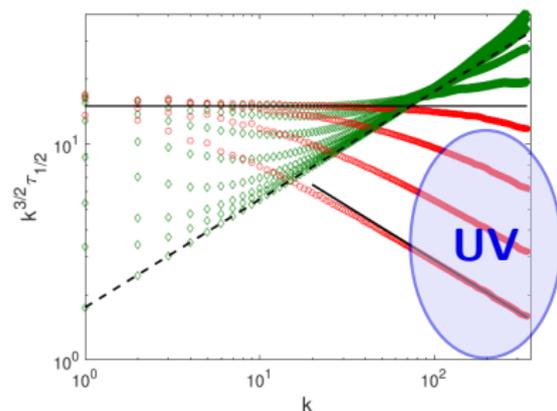
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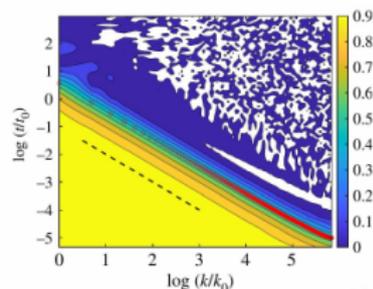
Edwards-Wilkinson ( $\lambda = 0$ ):  $z = 2$

# Results from numerical simulations

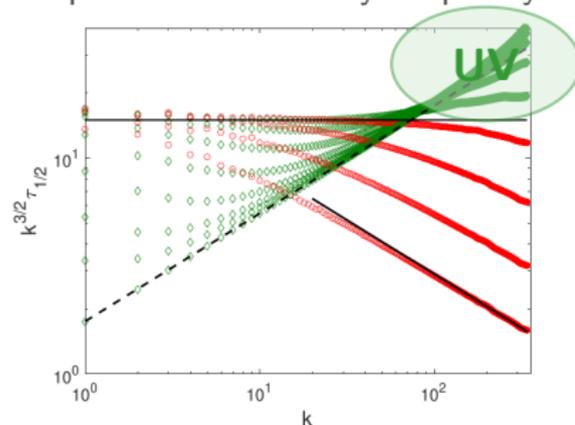
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compensated half decay frequency



Inviscid Burgers ( $\nu = 0$ ):  $z = 1$

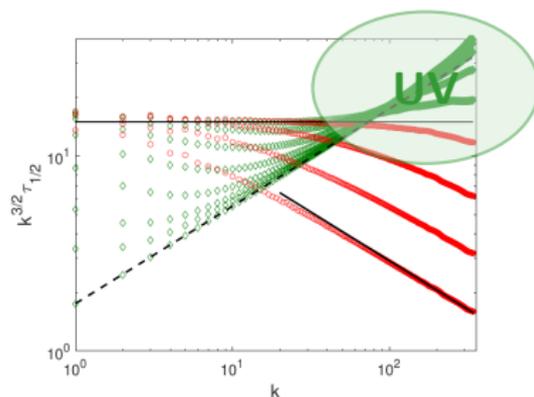
**unpredicted scaling regime !**

# The non-perturbative side of the KPZ equation (II)

- ▶ one coupling constant  $g = \lambda^2 D / \nu^3$

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \sqrt{D} \eta$$

- ▶ 1D KPZ-Burgers equation in the inviscid limit



▶ **inviscid limit**  $\nu \rightarrow 0 \iff$

**strong-coupling limit**  $g \rightarrow \infty$

non-perturbative feature arises also in  $d = 1$ !

# The Inviscid Burgers fixed point from Functional Renormalisation Group

► FRG with simplest approximation:  $\nu \longrightarrow \nu_{\kappa}$ ,  $D \longrightarrow D_{\kappa}$

▷ one effective coupling  $\hat{g}_{\kappa} = \kappa^{d-2} \lambda^2 D_{\kappa} / \nu_{\kappa}^3$

▷ define  $\hat{w}_{\kappa} = \hat{g}_{\kappa} / (1 + \hat{g}_{\kappa}) \in [0, 1]$

► FRG flow equation for  $\hat{w}_{\kappa}$  in  $d = 1$

$$\kappa \partial_{\kappa} \hat{w}_{\kappa} = \hat{w}_{\kappa} (3 - 2z_{\kappa}(\hat{w}_{\kappa})) (1 - \hat{w}_{\kappa})$$

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▶ 3 fixed point solutions

▶ EW:  $\hat{w}_{*} = 0$

$\chi = 1/2$ ,  $z_{EW} = 2$

UV stable

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▶ KPZ:  $0 < \hat{w}_* < 1$

$\chi = 1/2$ ,  $z_{\text{KPZ}} = 3/2$

IR stable

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IR stable

▶ IB:  $\hat{w}_* = 1$

$\chi = 1/2$ ,  $z_{\text{IB}} = 1$

UV stable

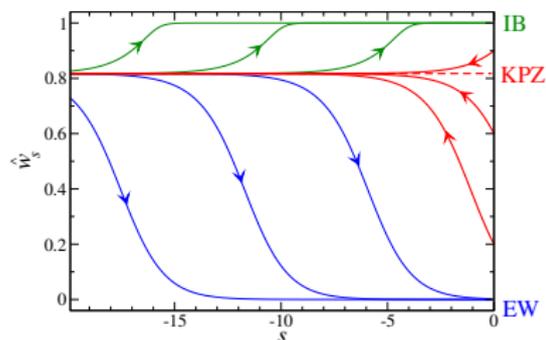
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▶ 3 fixed point solutions

▶ EW:  $\hat{w}_* = 0$

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←: IR flow

from small scales  $\kappa = \Lambda$

to large scales  $\kappa \rightarrow 0$

→: UV flows

backwards from  $\kappa \rightarrow 0$  to  $\kappa = \Lambda$

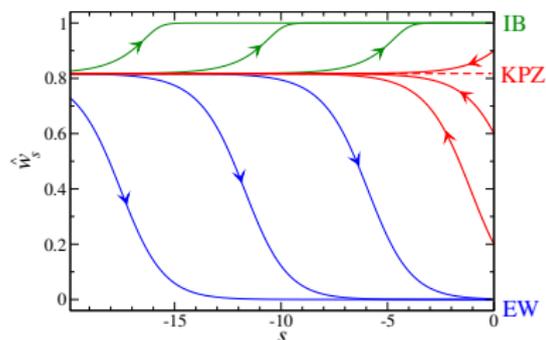
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▶ 3 fixed point solutions

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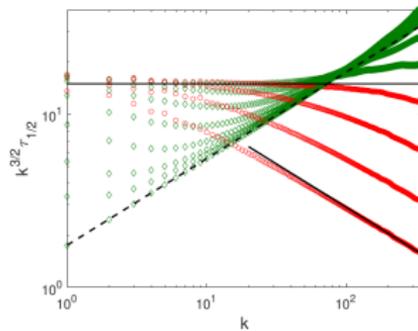
→: UV flows

backwards from  $\kappa \rightarrow 0$  to  $\kappa = \Lambda$

evidences the existence of the new IB fixed point in  $d = 1$  !



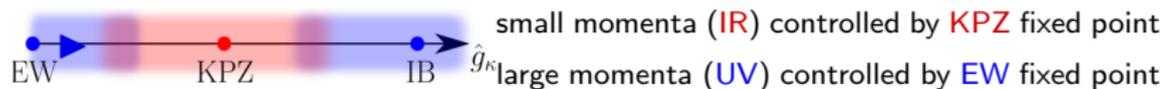
## Quantitative description ?



# The Inviscid Burgers fixed point from Functional Renormalisation Group

► FRG with functional approximation:  $\nu \rightarrow \nu_{\kappa}(\omega, \mathbf{p})$ ,  $D \rightarrow D_{\kappa}(\omega, \mathbf{p})$

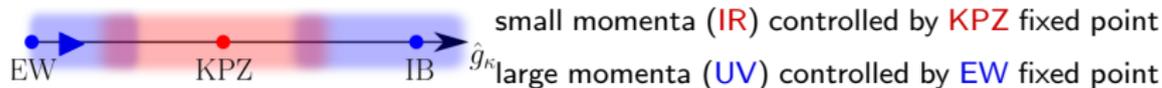
results for large viscosity in  $d = 1$



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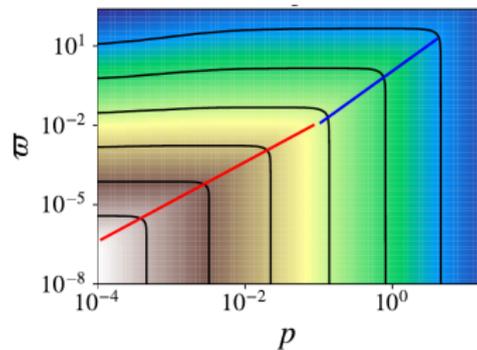
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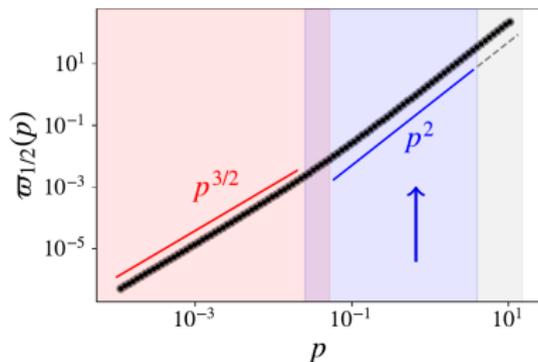
correlation function

$$C_\kappa(\omega, \mathbf{p}) = \frac{2D_\kappa(\omega, \mathbf{p})}{\omega^2 + (\mathbf{p}^2 \nu_\kappa(\omega, \mathbf{p}))^2}$$



half decay frequency  $\omega_{1/2}(\mathbf{p}) \sim p^z$

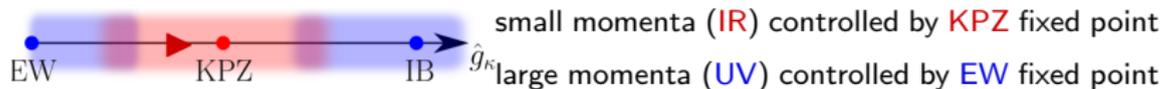
$$C(\omega_{1/2}(\mathbf{p}), \mathbf{p}) = C(0, \mathbf{p})/2$$



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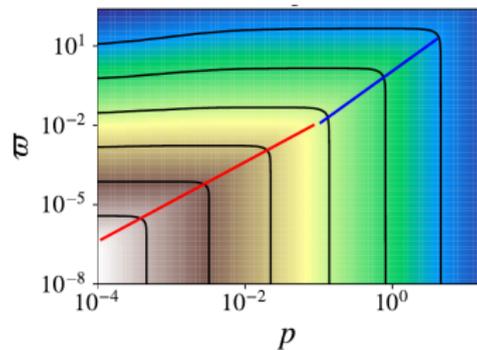
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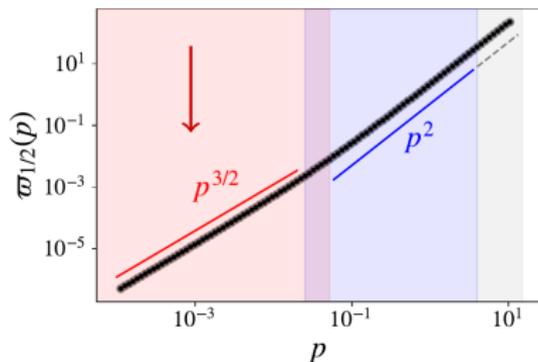
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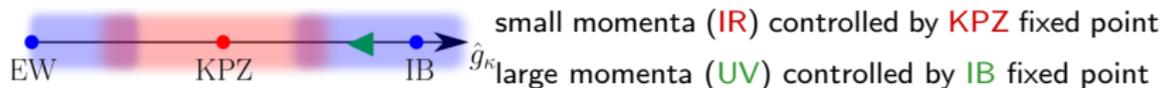
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# The Inviscid Burgers fixed point from Functional Renormalisation Group

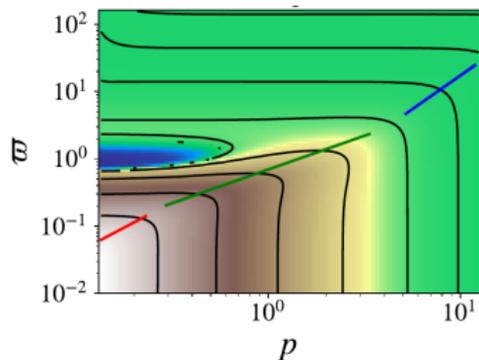
► FRG with functional approximation:  $\nu \longrightarrow \nu_\kappa(\omega, \mathbf{p})$ ,  $D \longrightarrow D_\kappa(\omega, \mathbf{p})$

results for small viscosity in  $d = 1$



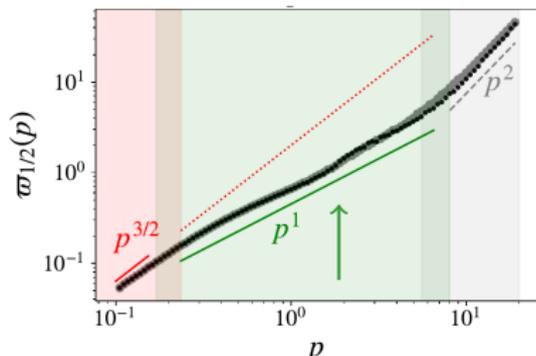
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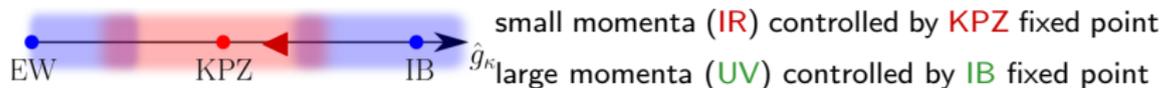
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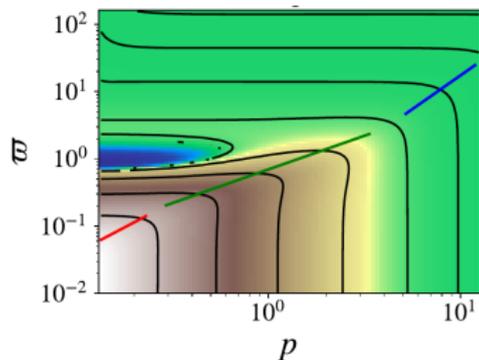
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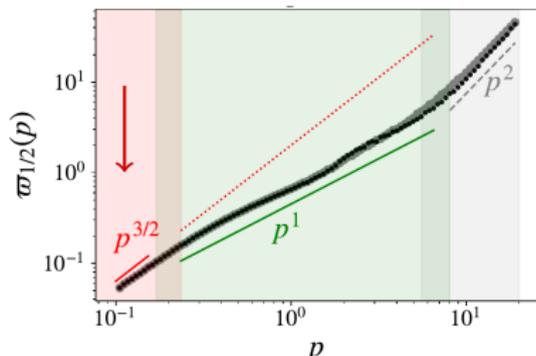
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half decay frequency  $\omega_{1/2}(p) \sim p^z$

$$C(\omega_{1/2}(p), \mathbf{p}) = C(0, \mathbf{p})/2$$



What about higher dimensions ?

# Functional Renormalisation Group: Existence of IB fixed-point in all $d$

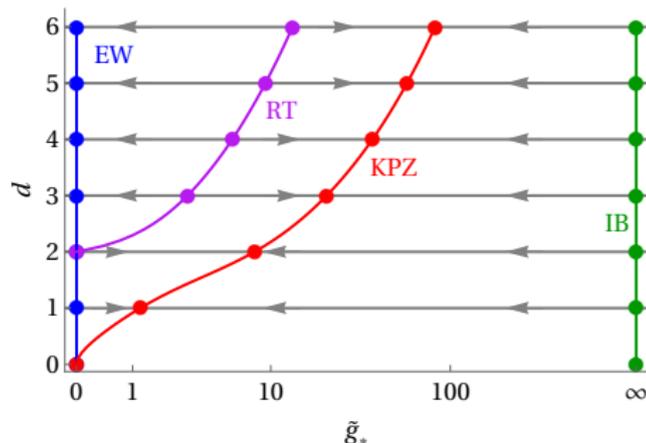
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▷ define  $\hat{w}_\kappa = \hat{g}_\kappa / (1 + \hat{g}_\kappa) \in [0, 1]$

► FRG flow equation for  $\hat{w}_\kappa$  in  $d > 1$

$$\partial_s \hat{w}_\kappa = 2 \hat{w}_\kappa (\chi_\kappa(\hat{w}_\kappa) + z_\kappa(\hat{w}_\kappa) - 2) (1 - \hat{w}_\kappa)$$



$z = 1$  scaling in all  $d$   
super-universal

Gosteva, Tarpin, Wschebor, LC

PRE 110 (2024)

Can we rigorously demonstrate that  $z = 1$  in all dimensions?

# Yes !

## From extended symmetries and large wavenumber limit

- ▶ **extended symmetry**: infinitesimal transformation such that the variation of the action is linear in the fields

⇒ yields exact functional Ward identities

### for Burgers action:

- time-gauged Galilean invariance:  $\mathcal{G} : \begin{cases} \mathbf{x} \rightarrow \mathbf{x} + \delta\mathbf{v}(t) \\ \mathbf{v} \rightarrow \mathbf{v} - \delta\dot{\mathbf{v}}(t) \end{cases}$
- time-gauged shift symmetry:  $\mathcal{R} : \begin{cases} \bar{v}(t, \mathbf{x}) \rightarrow \bar{v}(t, \mathbf{x}) + \delta\bar{v}(t) \\ + \text{other response fields} \end{cases}$

infinite set of exact Ward identities which relates any  $n$ -point vertex with one  $\mathbf{q} = 0$  to **lower-order** vertices

# Exact closure in the large wave-number limit

► flow for  $C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{x}_i\}) \equiv \left\langle v_{\alpha_1}(t_1, \mathbf{x}_1) \cdots v_{\alpha_n}(t_n, \mathbf{x}_n) \right\rangle_c$

$$\partial_{\omega_1} c_{\mathbf{k}_1}^{(1)} = -\frac{1}{2} \left( \text{loop diagram} \right) + \sum_{k+l=n} \left( \text{two-point diagram} \right)$$

exact (but infinite hierarchy of) flow

# Exact closure in the large wave-number limit

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$$\partial_\kappa c_c^{(n)} = -\frac{1}{2} \text{diagram} + \sum_{k+l=n} \text{diagram} \times \text{diagram}$$

The diagram on the left shows a vertex  $c_c^{(n)}$  with  $n$  external legs labeled  $\omega_1, \mathbf{k}_1, \dots$ . It is equal to  $-\frac{1}{2}$  times a diagram with a loop (labeled  $\omega, \mathbf{q}$  and  $-\omega, -\mathbf{q}$ ) and  $n$  external legs, plus a sum over  $k+l=n$  of two diagrams with  $k$  and  $l$  external legs respectively, connected by a red 'X'.

exact (but infinite hierarchy of) flow

large  $k_i$  →

$$\partial_\kappa c_c^{(n)} = -\frac{1}{2} \text{diagram} + \sum_{k+l=n} \text{diagram}$$

The diagram on the right is similar to the one on the left, but the loop is labeled  $q \approx 0$  and the two diagrams in the sum are crossed out with a red diagonal line.

asymptotic flow at large wavenumber

# Exact closure in the large wave-number limit

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The diagram shows a vertex with  $n$  external lines. The first term is a self-energy loop with a red 'x' on the loop. The sum term consists of two vertices connected by a line, with a red 'x' on the connecting line. The second vertex has  $l$  external lines.

exact (but infinite hierarchy of) flow

large  $k_i$

$$\partial_{\kappa} C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{x}_i\}) = -\frac{1}{2} \text{diagram} + \sum_{k+l=n} \text{diagram}$$

The diagram is similar to the exact flow equation, but the second term (the sum) is crossed out with a red diagonal line. The loop diagram now has a red 'x' and the label  $q \approx 0$  next to it.

asymptotic flow at large wavenumber

$$\partial_{\kappa} C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{x}_i\}) = \mathcal{K}^{(2)}(\{t_i, \mathbf{k}_i\}) C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{x}_i\}) + \mathcal{O}(k_{\max})$$

The diagram shows a vertex with  $n$  external lines, with  $\mathcal{K}^{(2)}$  written above it and  $\mathcal{O}(k_{\max})$  written to the right.

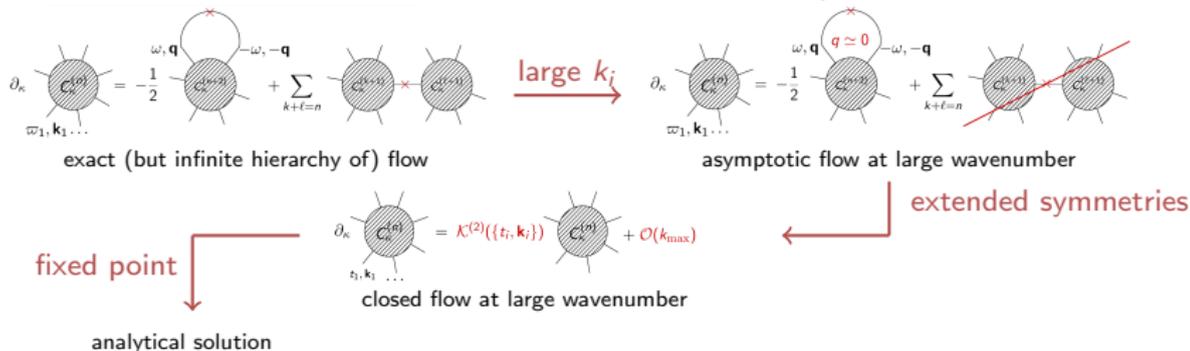
closed flow at large wavenumber

extended symmetries



# Exact closure in the large wave-number limit

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$$C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{x}_i\}) = C_{\alpha_1 \dots \alpha_n}^{(n)}(\{0, \mathbf{x}_i\}) \times \text{dominant term}$$

solution at the fixed point

$$C_{\alpha_1 \dots \alpha_n}^{(n)}(\{t_i, \mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2} \left| \sum_{\ell} \mathbf{k}_{\ell} t_{\ell} \right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \ll \tau \\ \exp\left(-\alpha_{\infty} \frac{L^2}{\tau} \left| t \sum_{k\ell} \mathbf{k}_k \cdot \mathbf{k}_{\ell} + \mathcal{O}(|\mathbf{k}_{\max}|L)\right)\right) & t_i \gg \tau \end{cases}$$

M. Tarpin, LC, N. Wschebor, Phys. Fluids 30 (2018), LC, J. Fluid Mech. Perspectives 950 (2022)

Fontaine, Vercesi, Brachet, LC, PRL 131 (2023), Gosteva, Tarpin, Wschebor, LC, PRE 110 (2024)

# Exact asymptotic solution at large wavenumbers for inviscid Burgers fixed point

- ▶ fixed-point solution at large  $\mathbf{p}$  (UV) from FRG:

$$C(t, \mathbf{p}) = C(0, \mathbf{p}) \times \begin{cases} \exp\left(-\alpha_0 (|\mathbf{p}|t)^2 + \mathcal{O}(|\mathbf{p}|L)\right) & t \ll \tau_0 \\ \exp\left(-\alpha_\infty |\mathbf{p}|^2 |t| + \mathcal{O}(|\mathbf{p}|L)\right) & t \gg \tau_0 \end{cases}$$

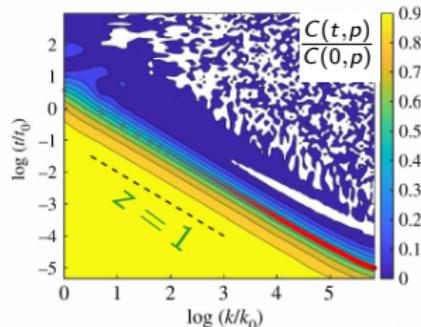
- proof of  $z = 1$  scaling at small  $t$  in all  $d$  ( $|\mathbf{p}|^z t \equiv |\mathbf{p}|t$ )
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- prediction of a crossover at large  $t$

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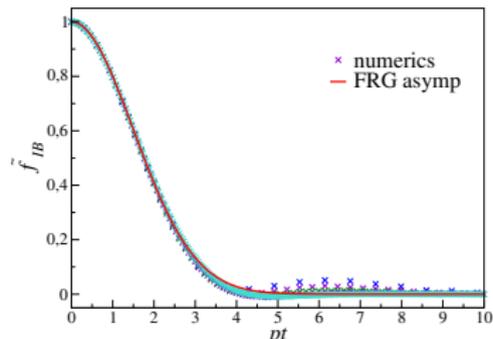
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Cartes *et al.*, Phil. Trans. A 380 (2022)



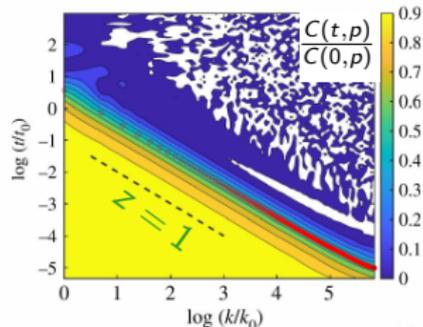
Fontaine, Vercesi, Brachet, LC, PRL 131 (2023)

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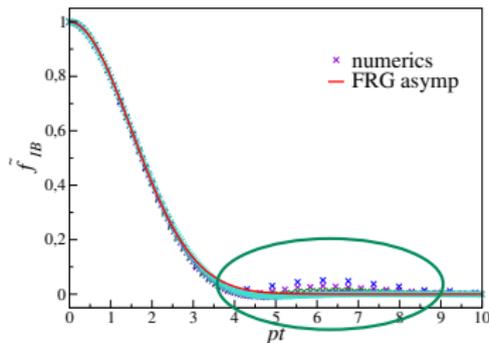
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existence of a new fixed point in the inviscid limit  
... only realisation ?

# Stochastic 3D Navier-Stokes equation

## Model A of Forster-Nelson-Stephen

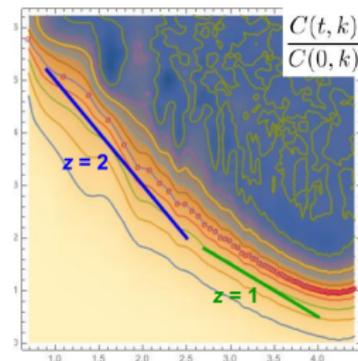
### ► Langevin dynamics for near-equilibrium fluids

Forster, Nelson, Stephen, PRL **36** (1976); Forster, Nelson, Stephen, PRA **16** (1977)

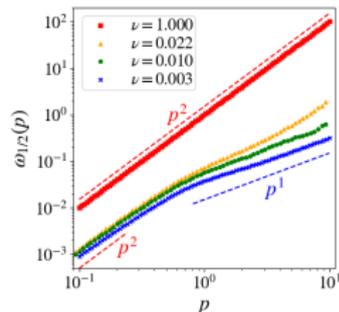
Bandak, Goldenfeld, Mailybaev, Eyink, PRE **105** (2022)

$$\partial_t \mathbf{v} + \lambda_0 (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \pi + \nu_0 \nabla^2 \mathbf{v} + \underbrace{\mathbf{f}}_{\text{thermal noise}}, \quad \nabla \cdot \mathbf{v} = 0$$

### numerics:



### FRG study:



# Realm of the Inviscid Burgers fixed point

**IB fixed point generically appears in :**

► Kuramoto-Sivashinsky equation

$$\partial_t h = \nu \nabla^2 h + \tau \nabla^4 h + \lambda (\nabla h)^2$$

■ microscopic scale:  $\nu < 0$

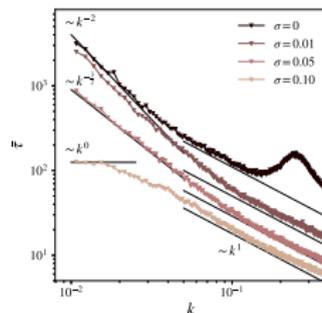
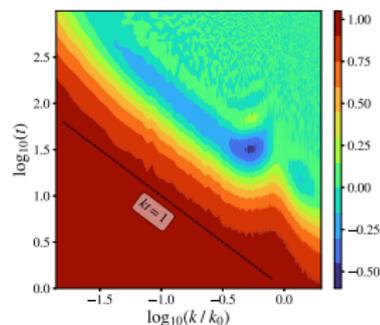
■ macroscopic scale: KPZ dynamics  $\nu_{\text{eff}} > 0$

⇒ intermediate scales:  $\nu_{\text{eff}} \simeq 0$  !

► complex Ginzburg-Landau equation

$$i \partial_t \psi = i \psi + (c_2 - i) |\psi|^2 \psi - (c_1 - i) \nabla^2 \psi$$

▷ dynamics of the phase  $\theta$  maps in the turbulence phase regime to the KS equation



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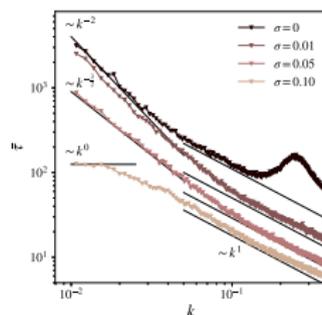
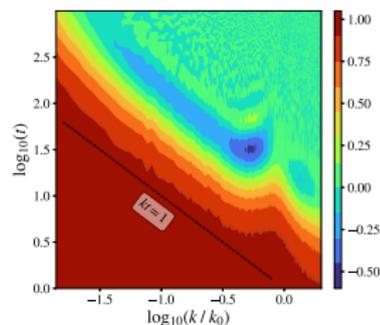
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**confirmed using FRG  
for Kuramoto-Sivashinsky equation**

Gosteva, Wschebor, LC, *in prep* (2026)

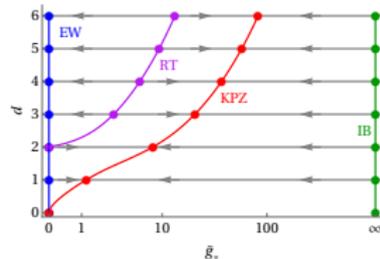


Vercesi, Poirier, Minguzzi, LC  
Phys. Rev. E **109** (2024)

# Summary and perspectives

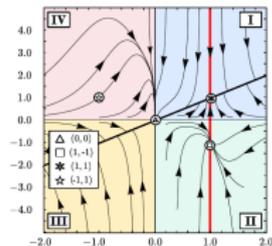
## Unpredicted Inviscid Burgers fixed point

- non-perturbative side of KPZ even in 1D
- exact closure of FRG equations based on extended symmetries  
 $\implies$  generic scaling  $z = 1$
- relevant in many situations



## Other variations on the same theme

- FRG for coupled KPZ equations
- FRG for conserved KPZ equation



Thank you for your attention !

