Liouville CFT
from probabilistic construction to bootstrap solution

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Based on joint works with C. Guillarmou, A. Kupiainen and V. Vargas
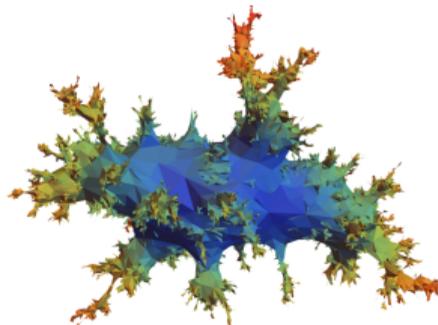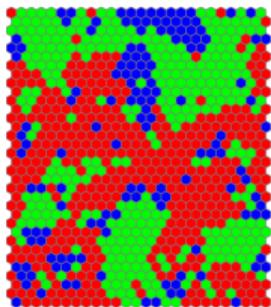
Context

Statistical physics model in 2D at criticality

Conformal Field Theories

Belavin-Polyakov-Zamolodchikov (1984): Conformal Bootstrap

Context

Discrete statistical physics models $\quad F \mapsto \sum_{\sigma \text{ config}} F(\sigma) e^{-H_\beta(\sigma)}$

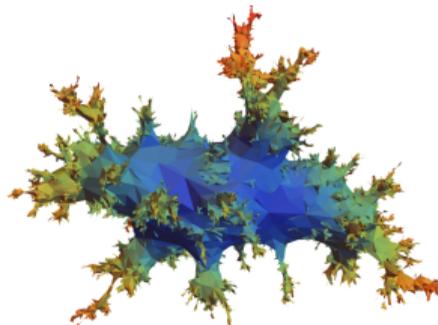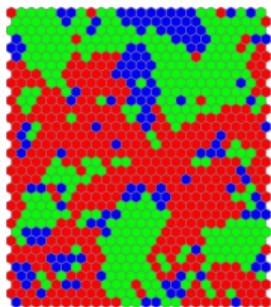Scaling limit $\quad F \mapsto \int F(\Phi) e^{-S(\Phi)} \, D\Phi =: \langle F \rangle$

Fields $x \in \Sigma \mapsto V_\alpha(\Phi, x) := V_\alpha(x)$ indexed by α

Correlation functions:

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_m}(x_m) \rangle$$

Context

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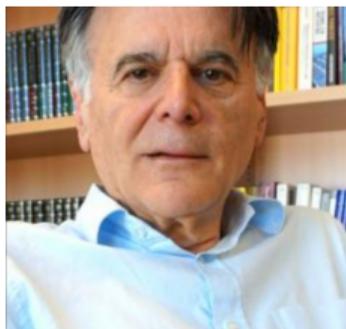

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Fields $x \in \Sigma \mapsto V_\alpha(\Phi, x) := V_\alpha(x)$ indexed by α

Correlation functions: CFT? (Gawedski, Segal, Borcherds-Frenkel,...)

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_m}(x_m) \rangle$$

" *The manuscript that follows was written fifteen years ago...I just wanted to justify my proposed definition by checking that all know examples of CFTs did fit the definition. This task held me up....*"

GRAEME SEGAL, *The definition of Conformal Field Theory* (2004).

This talk: Path integral and Conformal Bootstrap for Liouville CFT

Path integral for Liouville CFT

Riemann surface Σ, Riemannian metric g

$$F \mapsto \int F(\Phi) e^{-S_\Sigma(\Phi, g)} D\Phi$$

Liouville action

$$S_\Sigma(\Phi, g) = \frac{1}{4\pi} \int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + \mu e^{\gamma\Phi}) \mathrm{dv}_g$$

Parameters

$$\underline{\gamma \in (0, 2)}, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2} \quad , \mu > 0$$

Gaussian Free Field

Let X_g be the GFF on Σ in the metric g on Σ

$$X_g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \geq 1} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussians
- $(e_n)_n$ orthonormal basis of eigenfunctions of Laplacian Δ_g with eigenvalues $(\lambda_n)_n$ and b.c. $\int_\Sigma e_n \mathrm{dv}_g = 0$

Gaussian integral:

$$\int F(\Phi) e^{-\frac{1}{4\pi} \int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi = (\det'(\Delta_g)/\mathrm{v}_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c + X_g) \Big] \, dc$$

Liouville path integral (DGKRV 14-16')

$$\langle F \rangle_{\Sigma,g} := (\det'(\Delta_g)/v_g(\Sigma))^{-1/2} \int_{\mathbb{R}} \mathbb{E}\Big[F(c + X_g) e^{-\frac{1}{4\pi} \int_{\Sigma} (QK_g(c+X_g) + \mu e^{\gamma(c+X_g)}) dv_g} \Big] dc$$

where

- $\mu > 0$, $\gamma \in (0, 2)$ and $Q = 2/\gamma + \gamma/2$
- X_g be the GFF on Σ in the metric g
- $e^{\gamma X_g} dv_g$ is a random measure (Gaussian multiplicative chaos, Kahane '85)

$$e^{\gamma X_g(x)} dv_g(x) := \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\epsilon}(x)} dv_g(x)$$

with $X_{g,\epsilon}$ a regularization of X_g

Correlation functions

Set for $\alpha \in \mathbb{R}$ and $x \in \Sigma$

$$V_\alpha(x) := e^{\alpha \Phi(x)}$$

Correlation functions

$$\langle \prod_{j=1}^{m} V_{\alpha_j}(x_j) \rangle_{\Sigma, g}$$

Theorem (DGKRV '14 -'16): The correlation functions are non trivial iff the Seiberg bounds are satisfied

$$\forall j \quad \alpha_j < Q \qquad \text{and} \qquad \sum_{j=1}^{m} \alpha_j > \chi(\Sigma) Q$$

with $\chi(\Sigma)$ the Euler characteristics.

Geometric rules

▶ **Diffeomorphism invariance:** for $\psi : \Sigma \to \Sigma$ diffeo

$$\langle \prod_j V_{\alpha_i}(\psi(x_j)) \rangle_{\Sigma, g} = \langle \prod_j V_{\alpha_j}(x_j) \rangle_{\Sigma, \psi^* g}$$

▶ **Local scale invariance:** for $\varphi : \Sigma \to \mathbb{R}$ smooth

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma, e^\varphi g} = e^{\frac{c_L}{96\pi} \int_\Sigma |d\varphi|_g^2 + 2K_g \varphi} \left(\prod_i e^{-\Delta_{\alpha_i} \varphi(z_i)} \right) \langle \prod_i V_{\alpha_i}(z_i) \rangle_{\Sigma, g}$$

where

- $\mathbf{c}_L \in \mathbb{C}$ is the central charge
- conformal weight of the primary field V_α

$$\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$$

Riemann sphere and structure constants

3 point correlation function

$$\langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) \rangle_{\hat{\mathbb{C}}, g_0}$$

Structure constant

$$C(\alpha_1, \alpha_2, \alpha_3) := \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$

DOZZ formula (KRV '17)

$$C(\alpha_1, \alpha_2, \alpha_3) = (\pi \, \mu \, \ell(\frac{\gamma^2}{4}) \, (\frac{\gamma}{2})^{2-\gamma^2/2})^{\frac{2Q-\bar{\alpha}}{\gamma}}$$
$$\times \frac{\Upsilon'_{\frac{\gamma}{2}}(0) \Upsilon_{\frac{\gamma}{2}}(\alpha_1) \Upsilon_{\frac{\gamma}{2}}(\alpha_2) \Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2Q}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_1}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_2}{2}) \Upsilon_{\frac{\gamma}{2}}(\frac{\bar{\alpha}-2\alpha_3}{2})}$$

with

$$\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3, \quad \ell(x) = \Gamma(x)/\Gamma(1-x)$$

and the $\Upsilon_{\frac{\gamma}{2}}$ function defined as analytic continuation of the following integral defined for $0 < \Re(z) < Q$

$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z\right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4}) \sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

4 point correlation function (GKRV '20)

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}}$$

4 point correlation function (GKRV '20)

Theorem: assume

$$\alpha_1 + \alpha_2 > Q \quad \text{and} \quad \alpha_3 + \alpha_4 > Q.$$

Then for $|z| < 1$

$$
\begin{aligned}
&\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}} \\
&= \int_{\mathbb{R}_+} C(\alpha_1, \alpha_2, Q - ip) C(\alpha_3, \alpha_4, Q + ip) |z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_{(\Delta_{\alpha_i})_i, p}(z)|^2 dp
\end{aligned}
$$

where $\mathcal{F}_{(\Delta_{\alpha_i})_i, p}$ are the holomorphic conformal blocks.

Conformal blocks

The conformal blocks are holomorphic

$$\mathcal{F}_{(\Delta_{\alpha_i})_i, p}(z) = \sum_{n=0}^{\infty} \beta_n z^n$$

where

- $\beta_n = \sum_{|\nu|, |\bar{\nu}| = n} v(\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{Q+ip}, \nu) Q^{-1}_{\Delta_{Q+ip}}(\nu, \bar{\nu}) v(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{Q+ip}, \bar{\nu})$

- $\nu = (k_1 \geq k_2 \geq \dots)$ Young diagram of size $|\nu| = \sum_i k_i$

- $v(\Delta, \Delta', \Delta'', \nu) = \prod_j (k_j \Delta' + \Delta'' - \Delta + \sum_{u<j} k_u)$

- $Q_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ Shapovalov matrix with central charge $1 + 6Q^2$, and $Q^{-1}_{\Delta_{Q+iP}}(\nu, \bar{\nu})$ its inverse.

Crossing symmetry

Flip $\alpha_1 \leftrightarrow \alpha_3$ using the conformal map $z \mapsto 1 - z$

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}}} = \langle V_{\alpha_3}(0) V_{\alpha_2}(1-z) V_{\alpha_1}(1) V_{\alpha_4}(\infty) \rangle_{\hat{\mathbb{C}}}$$

Crossing symmetry

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Deduce

$$\int_{\mathbb{R}^+} C(\alpha_1, \alpha_2, Q - ip) C(\alpha_3, \alpha_4, Q + ip) |z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_1} - \Delta_{\alpha_2})} |\mathcal{F}_{(\Delta_{\alpha_i})_i, p}(z)|^2 dp$$

$$= \int_{\mathbb{R}^+} C(\alpha_3, \alpha_2, Q - ip) C(\alpha_1, \alpha_4, Q + ip) |1 - z|^{2(\Delta_{Q+ip} - \Delta_{\alpha_3} - \Delta_{\alpha_2})} |\tilde{\mathcal{F}}_{(\Delta_{\alpha_i})_i, p}(1-z)|^2 dp$$

with $\tilde{\mathcal{F}}_{(\Delta_{\alpha_i})_i, p}$ is obtained from $\mathcal{F}_{(\Delta_{\alpha_i})_i, p}$ by flipping $\alpha_1 \leftrightarrow \alpha_3$
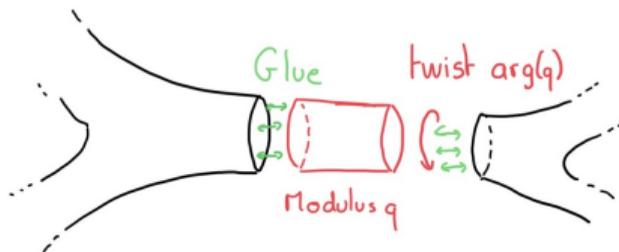
Conformal bootstrap in physics

- ► used to construct and solve CFTs
- ► correlation fcts assumed to have conformal bootstrap representation
- ► conformal blocks are known
- ► use crossing symmetry to find the structure constants
- ► BPZ (84') solved this way minimal models (e.g. Ising)
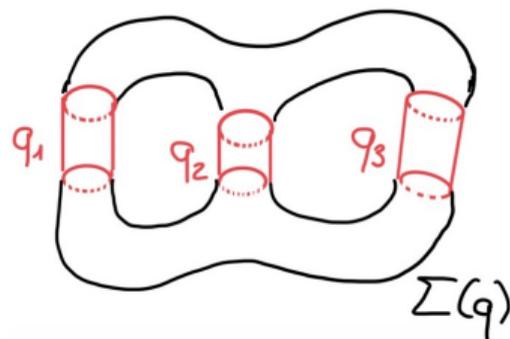
Moduli space and plumbing coordinates

▶ Pants

▶ Gluing annuli

Glue

twist arg(q)

Modulus q
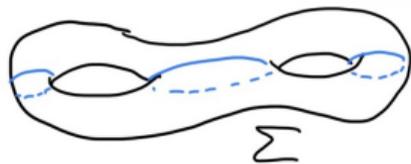
Local coordinates on $\mathcal{M}_{h,m}$

▶
$$q \in \mathbb{D}^{3h-3+m} \mapsto \Sigma(q)$$

q_1 q_2 q_3

$\Sigma(q)$

Conformal Bootstrap

Theorem (GKRV '21):

$$\langle V_{\alpha_1}(x_1) \ldots V_{\alpha_m}(x_m) \rangle_{\Sigma(q),g} = \int_{\mathbb{R}_+^{3\mathbf{h}-3+m}} \rho(p,\alpha) |\mathcal{F}_{c_L,p,\Delta_\alpha}(q)|^2 \mathrm{d}p$$

- $\rho(p,\alpha)$ product of structure constants

- $\mathcal{F}_{c_L,p,\Delta_\alpha}(q)$ local holomorphic series: conformal blocks

Applications or related results

▶ Conformal bootstrap for Liouville CFT on open surfaces.
  $\hookrightarrow$ random modulus of random planar maps on tori

▶ Representation of Mapping class group in the space of conformal blocks,
  Equivalent to Fock-Goncharov-Kashaev quantization of Teichmuller

▶ Integrability results for CLE or other Stats-Phys models

▶ Other CFTs (Toda, imaginary Liouville, WZW models)?